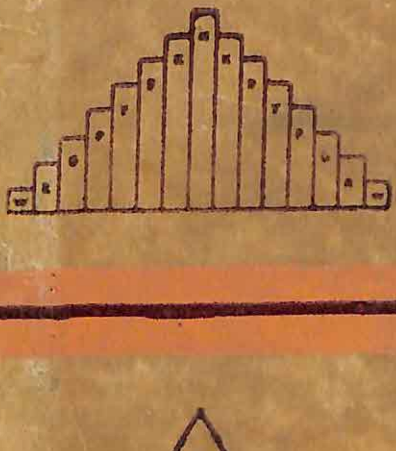
J. N. KAPUR

SUGGESTED EXPERIMENTS IN School Mathematics



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SUGGESTED EXPERIMENTS
IN
SCHOOL MATHEMATICS
[VOLUME III]

SUGGESTED EXPERIMENTS

IN

SCHOOL MATHEMATICS

Volume III

(EXPERIMENTS 301 To 500)

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PREFACE

This is the third volume of the series "SUGGESTED EXPERIMENTS IN SCHOOL MATHEMATICS" and contains Experiments 301—500. Volume I (Experiments 1—100) containing one hundred experiments was first published in 1967 and a revised edition has been brought out quite recently. Volume II (Experiments 101—300) was also published in March 1969.

The emphasis in the first volume was on 'activities'; in the second volume it was on 'patterns'; and in the third volume, it is on 'concepts'. A common objective of the three volumes is to give the teachers and students 'insight' into both the existing and the new school mathematics.

The first four chapters here deal with the structure of the system of natural numbers, integers, rational numbers and real numbers respectively. The fifth chapter deals with the properties of general binary operations. The sixth chapter deals with equivalent sentences including both equations and inequations.

Hindu-Arabic, Greek, Roman and Egyptian systems of numeration are dealt with in chapter 7 and the advantages of the Hindu-Arabic system are explicitly brought out there.

In chapter 8, some of the patterns of numbers considered in Volume II, are discussed further. The need for caution in inductive reasoning is pointed out and it is shown that the truth set of an open sentence depends very much on the replacement set under consideration.

Chapter 9 deals with H.C.F. and L.C.M. and tests of divisibility. Chapter 10 gives an alternative way of teaching multinomials.

The next chapter deals with the very important concept of graphs and shows how graphs can illuminate treatment of problems of arithmetic, geometry and nature.

The next six chapters deal with geometrical concepts. Chapter 12 deals with congruence mappings, chapter 13 deals with concepts of reflection, translation and rotation and their use in study of symmetry. Chapter 14 is concerned with the use of the concept of shearing in mensuration. Chapter 15 shows how three-dimensional geometry can be taught to students in the age group 10—12. Chapter 16 extends the concepts of plane geometry to the corresponding concepts of spherical geometry and these concepts are applied to problems of spherical astronomy. Applications of spherical geometry look 'realistic' to the students. Chapter 17 deals with topological concepts of graphs and networks.

The next four chapters are meant to illustrate the power of mathematics. Chapter 18 deals with combinatorial problem and probability. Chapter 19 shows how middle school mathematics can solve some really useful problems. Chapter 20 gives some more recent applications of mathematics to sequencing problem, transportation problem, game theory, inventory control and linear programming. Chapter 21 is aimed to bring home to the child the tremendous intellectual powers of mathematics.

The last chapter gives 50 puzzles and problems for brighter students and for motivating mathematical thinking.

It will thus be seen that both 'structures' and 'applications' are equally emphasized and it is shown how some of the so-called advanced ideas can be brought to the class room of the school.

This book can be used for self-study and for training programmes for those teachers who want to understand new mathematics and who want to enliven their teaching of their present curriculum. It can also be used by bright school students and by college under-graduates who missed these topics in their school days.

It is hoped that a fourth volume dealing with vectors, matrices, flow charts, motion geometry, non-Euclidean geometry, numerical methods, topology etc. and more detailed treatments of the topics discussed in the earlier volumes will be published some time in 1971.

I am grateful to all those who have encouraged me in writing these three volumes. I am grateful to my typist V. B. Pethkar for typing the manuscript, to Shri J. P. Gupta for seeing the book through the press and to my publishers for the pains they have taken.

All suggestions for improvement will be gratefully received.

I.I.T., Kanpur

J. N. KAPUR

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1

Structures of The Natural Number System

Experiments : 301 to 310

The children are all familiar with the system of Counting numbers or Natural numbers. The system is denoted by N. The children may not, however, be familiar with all the properties. They may be led to discover and verify the following properties.

Experiment 301. The Successor Natural Number.

(*i*) To every natural number, there is a 'next' natural number or a 'successor' natural number.

(*ii*) If the successor of n is denoted by n^*, then

$$\begin{aligned} n^* &= n+1 \\ n+m^* &= (n+m)^* \\ n\times m^* &= n\times m+n. \end{aligned}$$

(*iii*) Every number except '1' has a predecessor.

Experiment 302. Closure Properties in N.

(*i*) The sum of any two natural numbers is a natural number.

(*ii*) The product of any two natural numbers is a natural number.

(*iii*) If we subtract one natural number from another natural number, we *do not* always get a natural number.

(*iv*) If we divide one natural number by another natural number, we *do not* always get a natural number.

These four properties are expressed by saying *that the system of natural numbers is closed for the operations of addition and multiplication, but is not closed for the operations of division and subtraction.*

Experiment 303. Commutative Laws in N.

(*i*) For any two natural numbers a and b, $a+b=b+a$, *i.e.*, the sum is the same in whatever order we add the two natural numbers. This law is called the *commutative law for addition* in N.

(*ii*) For any two natural numbers a and b, $a\times b=b\times a$, *i.e.*, the product is the same in whatever order the numbers are multiplied. This law is called the *commutative law for multiplication* in N.

(*iii*) **Commutative law does not hold for subtraction** in N. In fact if $a-b$ is a natural number, then $b-a$ would not even be a natural number.

(*iv*) **Commutative law does not hold for division** in N. In fact if $a\div b$ is a natural number, then $b\div a$ would not even be a natural number.

Experiment 304. Associative Laws in N.

(*i*) For any three natural numbers a, b, and c, we have $(a+b)+c=a+(b+c)$ *i.e.*, sum is the same whether we associate a and b first and then associate c to the sum or we associate with a the sum of b and c. This law is called the *associative law for addition* in N.

(*ii*) For any three natural numbers a, b, c we have

$$(a\times b)\times c=a\times(b\times c).$$

This is called the *associative law for multiplication* in N.

(*iii*) **The associative law does not hold for subtraction** in N, *i.e.*,

$$(a-b)-c\neq a-(b-c)$$

for all a, b, c in N.

The students may construct examples where the L.H.S. is not a natural number, the R.H.S. is not a natural number, neither side is a natural number, both sides are natural numbers but are not equal. In fact in this case the two sides can never be equal. (Why ?)

(*iv*) **The associative law does not hold for division** in N, *i.e.*,

$$(a\div b)\div c\neq a\div(b\div c)$$

for all a, b, c in N.

The students may construct examples where the L.H.S. is not a natural number, the R.H.S. is not a natural number, neither side is a natural number, both sides give natural numbers and are not equal, both sides give natural numbers and are equal.

Experiment 305. Distributive Laws in N.

For any three natural numbers a, b, c, we have

(*i*) $a\times(b+c)=(a\times b)+(a\times c)$.

(*ii*) $(b+c)\times a=(b\times a)+(c\times a)$.

These are called *right and left distributive laws* respectively. These involve two operations, *viz.*, multiplication and addition and express the fact that *multiplication distributes addition* ($b+c$ is distributed into b and c).

Either of these laws can be deduced from the other law and the commutativity of multiplication.

We can also ask whether addition can distribute multiplication, *i.e.*, whether the following laws can be true for all a, b, c in N.

(*iii*) $a+(b\times c)=(a+b)\times(a+c)$ (not true).

(*iv*) $(b\times c)+a=(b+a)\times(c+a)$ (not true).

Thus addition does not distribute multiplication.

The system N is not closed for division and subtraction and therefore other distributive laws cannot hold. (See Experiment 346.)

Experiment 306. Order Structure in N.

(*i*) Given any two natural numbers a, b, we have one of the following three mutually exclusive possibilities :

$$a=b \text{ or } a<b \text{ or } b<a.$$

This is expressed by saying that the 'smaller than' relation in N satisfies the *trichotomy law*.

(*ii*) The 'greater than' relation in N also satisfies the trichotomy law *i.e.*, given any two natural numbers a and b, only one of the following three mutually exclusive possibilities holds ;

$$a=b \text{ or } a>b \text{ or } b>a.$$

(*iii*) The relations $>$, $<$, $\geqslant$, $\leqslant$ satisfy the *transitive law* in N, *i.e.*,

If $a>b$ and $b>c$, then $a>c$.

If $a<b$ and $b<c$, then $a<c$.

If $a \geqslant b$ and $b \geqslant c$, then $a \geqslant c$.
If $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$.

(*iv*) Relations which satisfy both trichotomy law and transitive law in N are called *order relations*. Both $>$ and $<$ are order relations and *we say that N has an order structure*.

Experiment 307. Consistency of Order Structure with Addition and Multiplication.

(*i*) If $a > b$, then whatever natural number c may be, $a+c > b+c$.
(*ii*) If $a > b$ and $c > d$, then $a+c > b+d$.
(*iii*) If $a > b$ then whatever natural number c may be, $ac > bc$.
(*iv*) If $a > b$ and $c > d$, then $a \times c > b \times d$.
Similar laws hold if '$>$' is replaced by '$<$'.

These laws express the fact that *in N the order structure is consistent with both addition and multiplication structures.*

Experiment 308. Cancellation Laws in N.

(*i*) If $a+c=b+c$, then $a=b$.
(*ii*) If $a \times c=b \times c$, then $a=b$.
(*iii*) If $c+a=c+b$, then $a=b$.
(*iv*) If $c \times a=c \times b$, then $a=b$.

These laws are expressed by saying that *in N the cancellation law holds for both addition and multiplication.*

Experiment 309. The Well-ordering Principle in N.

From the set N of natural numbers, we take any sub-set, finite or infinite. It will have a smallest member *e.g.*, the sub-set 1, 3, 5, 7 has a smallest member 1, sub-set 2, 4, 6, 8, ... of all even natural numbers has a smallest member 2.

Every finite sub-set will also have a greatest member, but an infinite sub-set will not have a greatest member.

The principle that every sub-set of N has always a smallest member is an important principle and is known as the *well-ordering* principle. It has important applications.

Experiment 310. Structures in N.

The system N of natural numbers has three types of structures *viz.*, additive, multiplicative and order. Both addition and multiplication are commutative and associative in N and the two structures are connected by the distributive laws. The order structure satisfies the trichotomy and transitive laws and this structure is consistent with both additive and multiplicative structures. In the next chapter, we shall discuss the structure of system I of integers. Some of the above laws will continue to hold, while others will break down and some new ones will be found to hold. Different systems differ in their laws. If all the laws of two given systems are the same, the two systems are essentially the same from the mathematical point of view. Later we shall study the structure of the system of rational numbers and again emphasize the similarities and differences from the laws for N and I.

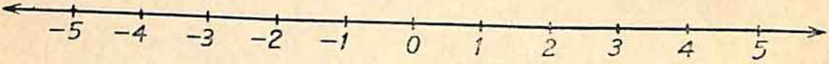
Before going to the next chapter, the readers may find it worthwhile to construct for themselves the structure of the system of whole numbers *i.e.*, the system consisting of natural numbers and zero on the same lines as given in this chapter for the system N.

2

Structures of The System of Integers

Experiments : 311 to 315

The system I of integers consists of the positive integers 1, 2, 3, 4, ……, negative integers, $-1, -2, -3, -4$, …… and the integer 0. The system can be represented on the number line as follows :



Experiment 311. Successors and Closure in I.

(*i*) For every integer there is both a successor integer and a predecessor integer. There is no exception in this case.

(*ii*) The sum, product and difference of two integers is always an integer, but the quotient of two integers need not be an integer. *The system I is therefore closed for addition, multiplication and subtraction, but is not closed for division.*

Experiment 312. Commutative, Associative and Distributive Laws in I.

(*i*) For any two arbitrary integers a, b, we have

$$a+b=b+a, \qquad a\times b=b\times a,$$
$$a-b\neq b-a, \qquad a\div b\neq b\div a.$$

Therefore, *in I, the commutative law holds for addition and multiplication, but it does not hold for subtraction and division.*

If $\quad a=b,$

then $\quad a-b=b-a$, and $a\div b=b\div a$

but for commutative law for subtraction and division, we want the result to be valid for *all* a and b.

(*ii*) For any three arbitrary integers a, b, c,

$$\begin{aligned}
&(a+b)+c=a+(b+c)\\
&(a\times b)\times c=a\times(b\times c)\\
&(a-b)-c\neq a-(b-c)\\
&(a\div b)\div c\neq a\div(b\div c) \qquad (b\neq 0,\ c\neq 0)
\end{aligned}$$

and so in I, the associative law holds for addition and multiplication, but does not hold for subtraction or division.

(*iii*) For any three integers, a, b, c,

$$\begin{aligned}
&a\times(b+c)=a\times b+a\times c\\
&(b+c)\times a=b\times a+c\times a\\
&a\times(b-c)=a\times b-a\times c\\
&(b-c)\times a=b\times a-c\times a
\end{aligned}$$

Thus multiplication distributes addition and subtraction both on the right and on the left.

Experiment 313. The Order Structure in I.

(*i*) In I also the trichotomy law holds both for $>$ and $<$ relations and the transitive law holds for $<$, $>$, $\leqslant$ and $\geqslant$.

(*ii*) If a, b, c, d are integers, then

(*a*) If $a>b$, then whatever integer c may be, $a+c>b+c$.

(*b*) If $a>b$ and $c>d$, then $a+c>b+d$.

(*c*) If $a>b$ and c is a positive integer, then $ac>bc$.
If $a>b$ and c is a negative integer, then $ac<bc$.
If $a>b$ and c is 0, then $ac=bc$.

(*d*) If $a>b$ and $c>d$, then $a\times c$ is not necessarily greater than $b\times d$, *e.g.*,

$$\begin{array}{rlrl}
3>2 & \text{and} & 5>4 & \text{but} \quad 15>8\\
3>2 & \text{and} & -4>-5 & \text{but} \quad -12\ngtr-10\\
-2>-3 & \text{and} & -4>-5 & \text{but} \quad 8\ngtr 15\\
-2>-3 & \text{and} & 7>4 & \text{but} \quad -14\ngtr-12
\end{array}$$

Corresponding results can be stated for '$<$'.

Experiment 314. Cancellation Laws in I and Absence of Well-ordering Principle in I.

(*a*) If a, b, c are integers, then

(*i*) If $a \pm c = b \pm c$, then $a = b$.

(*ii*) If $\pm c + a = \pm c + b$, then $a = b$.

(*iii*) If $a \times c = b \times c$ and $c \neq 0$, then $a = b$.

(*iv*) If $c \times a = c \times b$ and $c \neq 0$, then $a = b$.

(*b*) The well-ordering principle does not hold for the set I, *e.g.*, the sub-set

$$\ldots\ldots, -6, -4, -2, 0, 2, 4, 6, 8, \ldots\ldots$$

of I has no least member.

Experiment 315. Comparison of the Structures in N and I.

(*i*) So far the difference between the systems N and I lies in the following respects :

(*a*) I is closed for subtraction, while N is not.

(*b*) Multiplication distributes subtraction in I.

(*c*) If $a > b$, and c is any integer, then
$ac > bc$ if c is a positive integer,
$ac < bc$ if c is a negative integer,
$ac = bc$ if $c = 0$.

(*d*) In I, unlike in N, if $a > b$ and $c > d$, then ac is not necessarily greater than bd.

(*e*) Cancellation law for multiplication holds in I, provided the integer cancelled is not zero.

(*f*) The well-ordering principle does not hold in I.

There are some other points of difference which we note below :

(*ii*) In I there exists an integer which when added to any integer gives that integer itself ; this integer is obviously the integer 0 and is called the *additive identity* of the system of integers. In N, there is no such member. The system of whole numbers has however an additive identity.

(*iii*) In I, to every integer there exists another integer such that the sum of these two integers is the additive identity *viz.*, 0. Thus corresponding to 3, we have -3 and corresponding to -3 we have 3. Numbers like a and $-a$ which when added to each other give 0 are called *additive inverses*. In N the question of an additive inverse does not arise while in the system of whole numbers, the question makes sense but no number except 0 has an additive inverse.

(*iv*) Both in N and I, there exists a member which when multiplied by any number of the system gives that number of the system. This number is obviously the number '1'. This is called the *multiplicative identity* of both N and I.

(*v*) Both in N and I *multiplicative inverses* do not, in general, exist. Two numbers are said to be multiplicative inverses of each other, if their product is the multiplicative identity. In N the only member with a multiplicative inverse is '1' while in I there are two such members, *viz.*, '1' and '−1'.

(*vi*) Consider

$$\square + 4 = 3.$$

The truth set of this open sentence is null set if we consider N. However in I, the truth set is not null. In fact the system of integers was 'invented' by mathematicians to see that even such sentences have non-null truth sets. The open sentence

$$\square \times 4 = 3$$

has its truth set as null set both for N and I and the mathematicians 'invented' the system of rational numbers to see that even this type of open sentences have non-null truth sets.

(*vii*) The students may not be able to 'prove' all the laws stated above, but they can always 'verify' and it will be useful for them to verify all of them in a large number of cases.

3

Structures of The Rational Number System

Experiments : 316 to 331

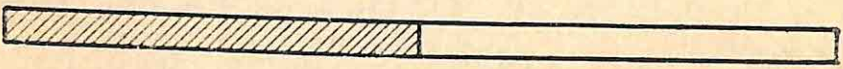
Experiment 316. The Need for Rational Numbers.

In the system of integers :
The truth set of $2 \times \square = 4$ is $\{2\}$
The truth set of $2 \times \square = -6$ is $\{-3\}$
The truth set of $2 \times \square = 5$ is ϕ
The truth set of $2 \times \square = -5$ is ϕ

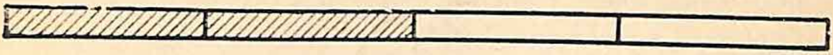
(since no integer multiplied by 2 gives 5 or -5). There is therefore need for 'inventing' new numbers which will be members of truth sets of such sentences. The need is similar to that for the truth sets of open sentences like $\square + 3 = 0$.

Experiment 317. Equivalent Fractions.

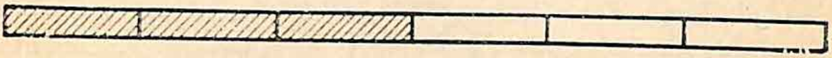
We divide the line into 2 equal parts and shade 1 of them.



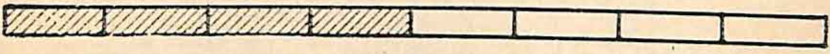
We divide the line into 4 equal parts and shade 2 of them.



We divide the line into 6 equal parts and shade 3 of them.



We divide the line into 8 equal parts and shade 4 of them.



In every case, we get the same shaded portion *viz.*, 'half'.

We express this by saying that

1 part out of 2=2 parts out of 4=3 parts out of 6=4 parts out of 8

$$\text{or } (1, 2)=(2, 4)=(3, 6)=(4, 8)=\ldots\ldots$$
$$\text{or } \tfrac{1}{2} = \tfrac{2}{4} = \tfrac{3}{6} = \tfrac{4}{8} = \ldots\ldots$$

In each case, the children can continue the pattern further to get, *e.g.*,

$$\tfrac{1}{2}=\tfrac{5}{10}=\tfrac{6}{12}=\ldots\ldots=\tfrac{10}{20}=\ldots\ldots=\tfrac{25}{50}=\ldots\ldots=\tfrac{35}{70}=\ldots\ldots$$

We call the numbers above the line the *numerators* and numbers below the line as *denominators*. The ratio of two natural numbers is called a *fraction*.

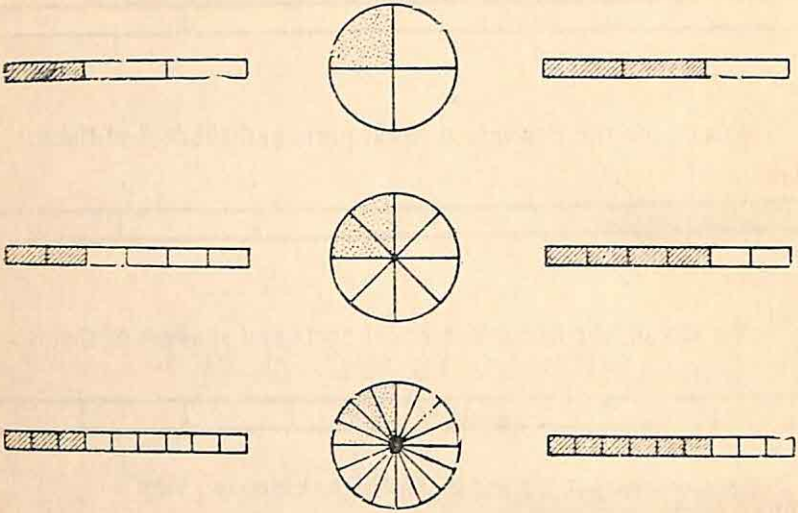
The children can discover the following three laws :—

(*i*) If we multiply the numerator and denominator by the same natural number, we get an equal fraction.

(*ii*) If we divide the numerator and denominator by a common factor of them, we again get an equal fraction.

(*iii*) If two fractions are equal, the cross products are equal, *i.e.*, product of numerator of the first and denominator of the second=product of numerator of the second and denominator of the first.

By using figures like those given below :



the children can get results like the following :

$$\frac{1}{3}=\frac{2}{6}=\frac{3}{9}=\frac{4}{12}=\frac{5}{15}=\ldots\ldots$$
$$\frac{1}{4}=\frac{2}{8}=\frac{3}{12}=\frac{4}{16}=\frac{5}{20}=\ldots\ldots$$
$$\frac{2}{3}=\frac{4}{6}=\frac{6}{9}=\frac{8}{12}=\frac{10}{15}=\ldots\ldots$$

and verify all the three laws stated on page 11.

The children can solve problems such as

$$\frac{1}{3}=\frac{4}{\square},\ \frac{\square}{6}=\frac{5}{10},\ \frac{3}{\square}=\frac{6}{8},\ \frac{5}{15}=\frac{25}{\square},$$
$$\frac{1}{2}=\frac{1\times 5}{2\times\square},\ \frac{30}{70}=\frac{30\div 5}{70\div\square},\ \ldots\ldots$$

Now $\frac{a}{b}=\frac{c}{d}$, if and only if $ad=bc$.

This can be used to verify that

(*i*) Every fraction is equal to itself, *i.e.*, $\frac{a}{b}=\frac{a}{b}$. (This is called reflexive property.)

(*ii*) If one fraction is equal to a second, then second fraction is equal to the first, *i.e.*, $\frac{a}{b}=\frac{c}{d}\Rightarrow\frac{c}{d}=\frac{a}{b}$. (This is called symmetric property.)

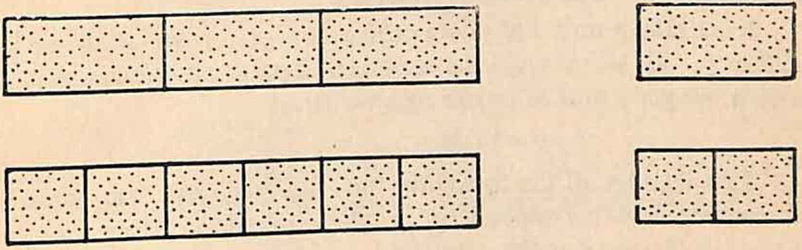
(*iii*) If first fraction is equal to a second fraction and the second fraction is equal to a third, then the first fraction is equal to the third fraction, *i.e.*, $\frac{a}{b}=\frac{c}{d}$, $\frac{c}{d}=\frac{e}{f} \Rightarrow \frac{a}{b}=\frac{e}{f}$. (This is called transitive property.)

Equality of fractions is a relation between fractions and the relation is reflexive, symmetric and transitive. Therefore, it is an equivalence relation. All fractions are divided into equivalence classes such that all fractions within a class are equal and no fraction of one class is equal to any fraction of another class, *e.g.*, each row below gives an equivalence class :

$$\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \ldots\ldots$$
$$\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \frac{5}{15}, \ldots\ldots$$
$$\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15}, \ldots\ldots$$

The fractions we have considered so far are those for which the numerator is less than the denominator. Such fractions are called *proper fractions*. *Improper fractions* are those for which the numerator is greater than the denominator.

To understand what is meant by $\frac{4}{3}$, we take one unit length, divide it into three equal parts and take four such parts.

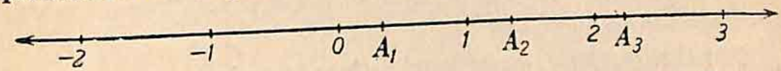


We easily get,

$$\frac{4}{3}=\frac{8}{6}=\frac{12}{9}=\frac{16}{12}=\ldots\ldots$$

and all the three laws stated above are easily verified.

Experiment 318. Representation of Fractions on the Number Line.



To represent fraction $\frac{1}{3}$, we divide the unit interval (0, 1) into three equal parts and take one to get the point A_1. We say that A_1

represents the fraction $\frac{1}{3}$. Similarly, A_2 represents $\frac{4}{3}$ or $1\frac{1}{3}$ and A_3 represents $2\frac{1}{5}$ or $\frac{11}{5}$. Numbers like $2\frac{1}{5}$ containing an integral part and a fractional part correspond to *mixed fractions*.

It is obvious that to every fraction there corresponds a unique point on the number line. What about the converse ?

If we divide the unit interval into three equal parts, we get points corresponding to improper fractions $\frac{1}{3}$ and $\frac{2}{3}$. If we divide first one-third interval into three equal parts, we get points corresponding to the improper fractions $\frac{1}{9}$, $\frac{2}{9}$. If we divide the first one-ninth interval into three equal parts, we get points corresponding to the improper fractions $\frac{1}{27}$, $\frac{2}{27}$. It is obvious that this process can be continued indefinitely and so there is a representation for an infinity of improper fractions on the number line and similarly there is a representation for an infinity of mixed fractions whose integral part is 1 or 2 or 3, and so on.

It may also be noted that the point A_1 represents not only fraction $\frac{1}{3}$ but also fractions $\frac{2}{6}$, $\frac{3}{9}$, $\frac{4}{12}$ etc., *i.e.*, the point corresponds to an equivalence class of fractions. Thus an equivalence class of fractions has a unique point on the number line coresponding to it and we say that the equivalence class represents one number.

Experiment 319. The Rational Numbers.

If we take a unit and take nothing out of it, we shall get the fraction $\frac{0}{1}$. If we divide the unit into 2 parts and taking nothing out of it, we get $\frac{0}{2}$ and so on, so that we get,

$$\frac{0}{1}=\frac{0}{2}=\frac{0}{3}=\frac{0}{4}=\ldots\ldots$$

This satisfies all the three laws for fractions and the point on the number line corresponding to the equivalence class of these fractions is the same as the point corresponding to the whole number 0. Similarly, the point corresponding to the class of equivalent fractions

$$\frac{1}{1}=\frac{2}{2}=\frac{3}{3}=\frac{4}{4}=\frac{5}{5}=\ldots\ldots$$

is the point corresponding to the whole number 1 and so on.

Can we have any class of fractions corresponding to negative integers on the number line ?

For this purpose, we 'create' rational numbers where a rational number $\frac{a}{b}$ is defined as the ratio of two integers a and b where

$b \neq 0$. We also want to identify rational numbers with fractions when a and b are positive integers. We would also like to extend the three laws of equality of fractions to continue to hold for rational numbers, *i.e.*, when we multiply the numerator and denominator of a rational number by a positive or negative integer, we should get an equal rational number. Thus

$$\frac{-a}{b} = \frac{a}{-b}$$

since the second ratio is obtained by multiplying the numerator and denominator of the first by (-1). We can define an equivalence relation in the class of rational numbers by stating that $\frac{a}{b} = \frac{c}{d}$, if and only if $ad = bc$. It is easily verified that this is an equivalence relation and gives rise to equivalence classes of the type

$$\ldots\ldots, \frac{-9}{-6}, \frac{-6}{-4}, \frac{-3}{-2}, \frac{3}{2}, \frac{6}{4}, \frac{9}{6}, \frac{12}{8}, \ldots\ldots$$

$$\ldots\ldots, \frac{-2}{-2}, \frac{-1}{-1}, \frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \frac{5}{5}, \ldots\ldots$$

$$\ldots\ldots, \frac{0}{-3}, \frac{0}{-2}, \frac{0}{-1}, \frac{0}{1}, \frac{0}{2}, \frac{0}{3}, \frac{0}{4}, \ldots\ldots$$

$$\ldots\ldots, \frac{3}{-3}, \frac{2}{-2}, \frac{1}{-1}, \frac{-1}{1}, \frac{-2}{2}, \frac{-3}{3}, \frac{-4}{4}, \ldots\ldots$$

$$\ldots\ldots, \frac{9}{-6}, \frac{6}{-4}, \frac{3}{-2}, \frac{-3}{2}, \frac{-6}{4}, \frac{-9}{6}, \ldots\ldots$$

Each of these equivalence classes of rational numbers determines one point on the number line.

We now have a set of rational numbers. We want now to study addition, multiplication, subtraction and division structures in it.

Experiment 320. Addition of Rational Numbers.

What can be the conditions which we would like the sum of rational numbers $\frac{a}{b}$ and $\frac{c}{b}$ to satisfy? Some of these are obviously the following:

(*i*) $\frac{a}{b} + \frac{c}{d}$ should depend on a, b, c, d.

(*ii*) $\frac{a}{b} + \frac{c}{d}$ should be a rational number.

(*iii*) $\frac{a}{b}+\frac{c}{d}$ should be same as the sum of two equivalent fractions $\frac{la}{lb}+\frac{mc}{md}$, where l and m are non-zero integers.

(*iv*) $\frac{a}{b}+\frac{c}{d}$ should be equal to $\frac{a}{1}+\frac{c}{1}$ or to $a+c$ when $b=d=1$, since $\frac{a}{1}$ and $\frac{c}{1}$ correspond to integers a and c and so $\frac{a}{1}+\frac{c}{1}$ should corsespond to $a+c$ and we identify $\frac{a}{1}$ and $\frac{c}{1}$ with a and c.

(*a*) Suppose we define $\frac{a}{b}+\frac{c}{d}=\frac{(a+c)}{(b+d)}$, it satisfies conditions (*i*), (*ii*) but does not satisfy conditions (*iii*) and (*iv*).

(*b*) Suppose we define $\frac{a}{b}+\frac{c}{d}=\frac{(a+c)}{bd}$, it satisfies conditions (*i*), (*ii*) and (*iv*), but does not satisfy condition (*iii*).

(*c*) Suppose we define $\frac{a}{b}+\frac{c}{d}=\frac{(ab+cd)}{bd}$. This also satisfies conditions (*i*), (*ii*) and (*iv*), but does not satisfy condition (*iii*).

(*d*) Suppose we define $\frac{a}{b}+\frac{c}{d}=\frac{(2ad+2bc)}{bd}$. This satisfies conditions (*i*), (*ii*) and (*iii*), but does not satisfy condition (*iv*).

(*e*) Suppose we define $\frac{a}{b}+\frac{c}{d}=\frac{\sqrt{ad}+\sqrt{cb}}{\sqrt{bd}}$. This satisfies conditions (*i*) and (*iii*) but does not satisfy conditions (*ii*) and (*iv*).

Let the children consider many more possibilities till they arrive at the definition $\frac{a}{b}+\frac{c}{d}=\frac{(ad+bc)}{bd}$, which satisfies all the four conditions laid down above. This is the definition we adopt.

The children may fail to find any other definition satisfying all the above four conditions, but does it prove that no other definition exists ?

Experiment 321. Additive Group Structure in Q.

We shall denote the set of rational numbers by Q where Q stands for the 'quotient' of two integers. We shall examine the various laws in $(Q, +)$.

(*i*) **Closure under Addition.** The students can easily verify that the sum of two rational numbers is a rational number.

(*ii*) **Associativity of Addition.** The students can easily verify by a number of examples that the addition of rational numbers is associative. Alternatively,

$$\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f}=\frac{ad+bc}{bd}+\frac{e}{f}$$
(using definition of addition of rational numbers)

$$=\frac{(ad+bc)\,f+bde}{bdf}$$
(using definition of addition of rational numbers)

$$=\frac{adf+(bcf+bde)}{bdf}$$
(using distributivity and associativity in system of integers)

$$=\frac{adf}{bdf}+\left(\frac{bcf}{bdf}+\frac{bde}{bdf}\right)$$
(using distributivity and associativity in system of integers)

$$=\frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right)$$
(using definition of equal fractions, cancellation law for fractions)

This establishes the associative law for addition of rational numbers. It may be noted that in proving this law, we have made use of (*i*) the associative law for addition of integers, (*ii*) distributive law of multiplication over addition for integers, (*iii*) definition of addition of fractions, and (*iv*) cancellation law for fractions. It may also be noted that here $b\neq0$, $d\neq0$, $f\neq0$, $bd\neq0$, $bf\neq0$.

(*iii*) **Existence of an Additive Identity.** This rational number is $\frac{0}{1}$, since

$$\frac{a}{b}+\frac{0}{1}=\frac{a.1+b.0}{b.1}=\frac{a}{b}$$

and $$\frac{0}{1}+\frac{a}{b}=\frac{0.b+1.a}{1.b}=\frac{a}{b}.$$

(*iv*) **Existence of an Additive Inverse.** For every rational number $\frac{a}{b}$, there exists a rational number $\frac{-a}{b}$, such that the sum of these two numbers is zero, since

$$\frac{a}{b}+\frac{-a}{b}=\frac{ab+(-a)b}{b}=\frac{0}{b}=\frac{0}{1}$$

$$\frac{-a}{b}+\frac{a}{b}=\frac{(-a)\,b+ab}{b}=\frac{0}{b}=\frac{0}{1}.$$

For this reason we also write $\frac{-a}{b}$ as $-\left(\frac{a}{b}\right)$, since it is the additive inverse of $\frac{a}{b}$.

(*v*) **Commutativity of Addition.** We have

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$$

$$\frac{c}{d}+\frac{a}{b}=\frac{cb+da}{db}$$

and the commutative law for rational numbers follows from the commutative law for integers.

Any set which satisfies properties (*i*)—(*iv*) is called an additive group. A set which satisfies properties (*i*)—(*v*) is called a commutative additive group. Thus the set of rational numbers is a commutative additive group.

Experiment 322. Multiplication of Rational Numbers.

We want to define $\left(\frac{a}{b}\right)\times\left(\frac{c}{d}\right)$ in such a way that

(*i*) $\left(\frac{a}{b}\right)\times\left(\frac{c}{d}\right)$ depends on a, b, c, d.

(*ii*) $\left(\frac{a}{b}\right)\times\left(\frac{c}{d}\right)$ is a rational number.

(*iii*) $\left(\frac{a}{b}\right)\times\left(\frac{c}{d}\right)$ is the same as $\left(\frac{la}{lb}\right)\times\left(\frac{mc}{md}\right)$.

(*iv*) $\left(\frac{a}{b}\right)\times\left(\frac{c}{d}\right)=\frac{ac}{1}$ when $b=1$, $d=1$.

The children can try various possible definitions, *e.g.*,

$$\frac{a}{b}\times\frac{c}{d}=\frac{(a+c)}{bd}$$

$$\frac{a}{b}\times\frac{c}{d}=\frac{ad}{bc}$$

$$\frac{a}{b}\times\frac{c}{d}=\frac{\sqrt{ac}}{\sqrt{bd}}$$

and see that the definition

$$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$$

satisfies all the four conditions. Can they show that this is the only definition which will satisfy all the four conditions ?

Experiment 323. Possibility of Multiplicative Group Structure in Rational Numbers.

Let the children examine whether the five properties which hold for addition in Q, also hold for the multiplication operation.

(*i*) **Closure for multiplication.** This is satisfied since the product of two rational numbers is also a rational number.

(*ii*) **Associativity of multiplication.** Associative law for multiplication holds since for every three rational numbers

$$\begin{aligned}
\frac{a}{b}\times\left(\frac{c}{d}\times\frac{e}{f}\right)&=\frac{a}{b}\times\left(\frac{ce}{df}\right) && \text{(definition of multiplication)}\\
&=\frac{a(ce)}{b(df)} && \text{(definition of multiplication)}\\
&=\frac{(ac)e}{(bd)f} && \text{(associativity of multiplication in integers)}\\
&=\frac{ac}{bd}\times\frac{e}{f} && \text{(definition of multiplication)}\\
&=\left(\frac{a}{b}\times\frac{c}{d}\right)\times\frac{e}{f} && \text{(definition of multiplication)}
\end{aligned}$$

Thus the associativity for multiplication of rational numbers is a consequence of the associativity of multiplication for integers.

(*iii*) **Existence of multiplicative identity.** There exists a multiplicative identy *viz.*, $\frac{1}{1}$ since for every rational number $\frac{a}{b}$

$$\left(\frac{a}{b}\right)\times\left(\frac{1}{1}\right)=\frac{(a\times 1)}{(b\times 1)}=\frac{a}{b};\ \left(\frac{1}{1}\right)\times\left(\frac{a}{b}\right)=\frac{(1\times a)}{(1\times b)}=\frac{a}{b}.$$

(*iv*) **Existence of multiplicative inverse.** Except for one rational number *viz.*, $\frac{0}{1}$, every other rational number $\frac{a}{b}$ has a multiplicative inverse *viz.*, $\frac{b}{a}$ since $\frac{a}{b}\times\frac{b}{a}=\frac{b}{a}\times\frac{a}{b}=\frac{ab}{ab}=\frac{1}{1}$.

(*v*) **Commutativity of multiplication.** The commutative law for rational numbers follows from the commutative law of integers, since

$$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}=\frac{ca}{db}=\frac{c}{d}\times\frac{a}{b}\,.$$

Due to the exception in (*iv*), the set of rational numbers does not form a multiplicative group. However, we have the following results which can be easily verified by the children.

(*a*) The set Q_0 of non-zero rational numbers forms a commutative multiplicative group.

(*b*) The set Q_+ of positive rational numbers forms a commutative multiplicative group.

Experiment 324. Field Structure in Rational Numbers.

We have seen the following properties for rational number system Q.

(*i*)—(*v*) (Q, +) is a commutative group.
(*vi*) Q is closed for multiplication.
(*vii*) Associative law for multiplication holds in Q.
(*viii*) There exists a multiplicative identity in Q.
(*ix*) Every non-zero element of Q has a multiplicative inverse.
(*x*) Commutative law for multiplication holds in Q.

Besides these ten, the following two distributive laws can also be seen to be true :

(*xi*)
$$\frac{a}{b}\times\left(\frac{c}{d}+\frac{e}{f}\right)=\frac{a}{b}\times\left(\frac{cf+de}{df}\right)=\frac{a(cf+de)}{bdf}$$
$$=\frac{acf}{bdf}+\frac{ade}{bdf}=\frac{ac}{bd}+\frac{ae}{bf}$$
$$=\left(\frac{a}{b}\right)\times\left(\frac{c}{d}\right)+\left(\frac{a}{b}\right)\times\left(\frac{e}{f}\right).$$

(*xii*)
$$\left(\frac{c}{d}+\frac{e}{f}\right)\times\left(\frac{a}{b}\right)=\left(\frac{cf+de}{df}\right)\left(\frac{a}{b}\right)=\frac{(cf+de)a}{bdf}$$
$$=\frac{(cfa+dea)}{bdf}=\frac{ca}{db}+\frac{ea}{fb}$$
$$=\left(\frac{c}{d}\right)\times\left(\frac{a}{b}\right)\times\left(\frac{e}{f}\right)\times\left(\frac{a}{b}\right).$$

In the proofs of these distributive laws, we use the distributive laws for integers.

A set with two operations which satisfies the above twelve laws is called a field. Thus (Q, +, ×) is a field.

The students may show that the following are not fields :

(N, +, ×), (I, +, ×), (Q_+, +, ×), (Q, −, ×), (Q_0, +, ×).

In (I, +, ×) all the above laws except (*ix*) hold.

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A system with the eleven properties [*i.e.*, with the exception of (*ix*) above] is called a commutative ring with unity. Thus (I, +, ×) is a commutative ring with unity. The system will be called a ring even if (*viii*), (*ix*) and (*x*) do not hold.

Experiment 325. Order Structure in Q.

A rational number is said to be greater than or less than another rational number according as it is on the right or the left of the other on the number line.

Alternatively, $\frac{a}{b} > \frac{c}{d}$ if $\frac{a}{b} - \frac{c}{d}$ is a positive rational number where $\frac{a}{b} - \frac{c}{d}$ is defined to be the sum of $\frac{a}{b}$ and the additive inverse of $\frac{c}{d}$.

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \left(-\frac{c}{d}\right) = \frac{a}{b} + \frac{-c}{d} = \frac{ad+b(-c)}{bd} = \frac{ad-bc}{bd}$$

$\frac{a}{b} > \frac{c}{d}$ if $ad-bc$ and bd have same sign.

$\frac{a}{b} = \frac{c}{d}$ if $ad-bc=0$.

$\frac{a}{b} < \frac{c}{d}$ if $ad-bc$ and bd have opposite signs.

Given any three rational numbers $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$,

(*i*) either $\frac{a}{b} > \frac{c}{d}$ or $\frac{a}{b} = \frac{c}{d}$ or $\frac{a}{b} < \frac{c}{d}$. (trichotomy law)

(*ii*) $\frac{a}{b} > \frac{c}{d}$ and $\frac{c}{d} > \frac{e}{f} \Rightarrow \frac{a}{b} > \frac{e}{f}$. (transitive law)

(*iii*) $\frac{a}{b} > \frac{c}{d} \Rightarrow -\left(\frac{a}{b}\right) < -\left(\frac{c}{d}\right)$.

(*iv*) $\frac{a}{b} > \frac{c}{d} \Rightarrow \frac{a}{b} + \frac{e}{f} > \frac{c}{d} + \frac{e}{f}$.

(*v*) $\frac{a}{b} > \frac{c}{d} \Rightarrow \frac{a}{b} \times \frac{e}{f} > \frac{c}{d} \times \frac{e}{f}$ if $\frac{e}{f}$ is positive.

(*vi*) $\frac{a}{b} > \frac{c}{d} \Rightarrow \frac{a}{b} \times \frac{e}{f} < \frac{c}{d} \times \frac{e}{f}$ if $\frac{e}{f}$ is negative.

The last three show the consistency of order structure with addition and multiplication structures.

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All these laws are obvious from the geometrical representation on the number line.

(*i*) states that given any two points P, Q on the number line, either P is on the right of Q or P coincides with Q or P is on the left of Q.

(*ii*) states that if P is on the right of Q and Q is on the right of R, then P must be on the right of R.

(*iii*) states that if P is on the right of Q, then its image point P′ must be on the left of the image point Q′.

(*iv*) states that if P is on the right of Q, then it remains on the right if P and Q are both translated through the same distance either towards the right or towards the left.

(*v*) states that if P is on the right of Q and OP and OQ are both magnified or contracted in the same ratio, the new position of P will still be on the right of Q.

(*vi*) states a law which can be obtained by combining (*iii*) and (*v*) *i.e.*, that if P is on the right of Q and OP and OQ are magnified or contracted in the same ratio and then the image points of the new points are taken, then the final image position of P is on the left of the first image position of Q.

Alternatively all these can be proved analytically from the definitions. Thus from (*iv*),

$$\left(\frac{a}{b}+\frac{e}{f}\right)-\left(\frac{c}{d}+\frac{e}{f}\right)=\frac{a}{b}-\frac{c}{d}$$

so that if the R.H.S. rational number is positive, so is the left hand fractional number.

Again from (*vi*)

$$\frac{a}{b}\times\frac{e}{f}-\frac{c}{d}\times\frac{e}{f}=\left(\frac{a}{b}-\frac{c}{d}\right)\left(\frac{e}{f}\right).$$

If $\frac{a}{b}>\frac{c}{d}$ and $\frac{e}{f}$ is negative, the L.H.S. is negative.

Experiment 326. Dense-pecking of the Set of Rationals.

Let $\frac{a}{b}$ and $\frac{c}{d}$ be two rational numbers and let $\frac{a}{b}>\frac{c}{d}$. Consider another rational number $\frac{e}{f}$ where

$$\frac{e}{f}=\frac{m\left(\frac{a}{b}\right)+n\left(\frac{c}{d}\right)}{m+n},$$

where m and n are natural numbers.

$$\frac{a}{b}-\frac{e}{f}=\frac{n}{m+n}\left(\frac{a}{b}-\frac{c}{d}\right),\ \frac{e}{f}-\frac{c}{d}=\frac{m}{m+n}\left(\frac{a}{b}-\frac{c}{d}\right).$$

Thus $\frac{a}{b}>\frac{c}{d}\Rightarrow\frac{a}{b}>\frac{e}{f}$ and $\frac{e}{f}>\frac{c}{d}$, so that $\frac{e}{f}$ is a rational number lying between $\frac{a}{b}$ and $\frac{c}{d}$. Since m, n can take an infinity of values, we get an infinity of rational numbers between $\frac{a}{b}$ and $\frac{c}{d}$, however near $\frac{a}{b}$ and $\frac{c}{d}$ may be. This is expressed by saying that the set Q is densely pecked. Given any rational number, we cannot say which is the next greater rational number, since whatever rational number we name, there will be an infinity of rational numbers below it and greater than the given number. The sets N and I are not densely pecked, for given any member of these sets, we can talk of its successor and between any two members of either set, there is a finite number of members of that set.

Experiment 327. Subtraction and Division Operations in Q.

Let the children verify or prove the following and similar other properties :

(*i*) Q is closed for subtraction.

(*ii*) Subtraction is not associative in Q, *i.e.*,

$$\frac{a}{b}-\left(\frac{c}{d}-\frac{e}{f}\right)\neq\left(\frac{a}{b}-\frac{c}{d}\right)-\frac{e}{f}.$$

(*iii*) There does not exist a subtractive identity :

$$\frac{a}{b}-\frac{o}{b}=\frac{a}{b}\ ;\ \frac{o}{b}-\frac{a}{b}\neq\frac{a}{b}.$$

(*iv*) The operation of subtraction is not commutative.

(*v*) Multiplication distributes subtraction in Q.

(*vi*) Q_0 is closed for division where division by $\frac{a}{b}$ is defined as the multiplication by the multiplicative inverse $\frac{b}{a}$ of $\frac{a}{b}$.

(*vii*) Associative law does not hold for division in Q_0.

(*viii*) Commutative law does not hold for division.

(*ix*) Division distributes addition on the left in Q.

(*x*) Division distributes subtraction on the left.

(*xi*) (Q, −, ×) is not a field.

Experiment 328. Isomorphism of the System of Integers with a Sub-System of Rational Numbers.

Consider the set Q of all rational numbers and the sub-set Q_1 of the rational numbers which consists of all rational numbers with denominator unity. Let the following correspondence between Q_1 and I be set

$$\frac{a}{1} \leftrightarrow a.$$

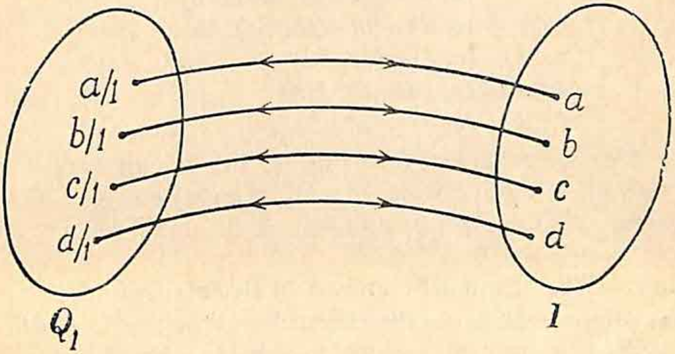
To every rational number $\frac{a}{1}$ belonging to Q_1, there corresponds a unique integer a belonging to I and to every integer a belonging to I, there corresponds a unique rational number $\frac{a}{1}$ belonging to Q_1. Thus the correspondence between Q_1 and I is one-one. However, it is not only one-one, it does something more ; it preserves addition and multiplication structures. What this means is the following :

If $\frac{a}{1} \leftrightarrow a$, $\frac{b}{1} \leftrightarrow b$,

then $\frac{a}{1}+\frac{b}{1} \leftrightarrow a+b$ and $\frac{a}{1}\times\frac{b}{1} \leftrightarrow a+b$,

so that the sum and product of two rational numbers belonging to Q_1 correspond to the sum and product of the corresponding integers and *vice versa*.

A one-one mapping which preserves certain structure or structures in the above sense is called isomorphism (iso = same, morphism = form). The two systems Q_1 and I are not identical, but they are almost the same in that though they have different members, these members are added and multiplied in the same way.



In mathematics, the concept of isomorphism is very important, because in mathematics we are more interested in the structure and relations between elements rather than in the names of elements or the form in which they are written.

Thus the system of rational numbers contains a sub-system which is isomorphic to the system of integers. This is partly expressed (not very accurately, of course) by saying that the set of integers is a subset of the set of rational numbers.

Similarly, the system of integers contains a sub-system (that of positive integers) which is isomorphic to the system of natural numbers. The correspondence is $+a \leftrightarrow a$. Let the children establish this. We again express this partly by saying that the set of natural numbers is a sub-set of the set of integers.

We may note here the distinction that is made sometimes between the set Q of rational numbers and the system (Q, +, ×) of rational numbers. A system is a set together with structures like these given by operation of addition, multiplication and order. However, when we talk of the sets of natural numbers or of integers or of rational numbers, we implicitly mean that we are talking about these systems.

Experiment 329. Another Notation for Rational Numbers.

A rational number $\frac{a}{b}$ can also be denoted by the ordered pair (a, b) of integers. We shall then have

$$\begin{aligned}
&(a, b)=(c, d) \text{ if and only if } ad=bc \\
&(a, b)+(c, d)=(ad+bc, bd) \quad b\neq 0, d\neq 0 \\
&(a, b)\times(c, d)=(ac, bd) \quad b\neq 0, d\neq 0 \\
&(a, b)\div(c, d)=(ad, bc) \quad b\neq 0, d\neq 0, c\neq 0 \\
&(a, b)-(c, d)=(ad-bc, bd) \quad b\neq 0, d\neq 0
\end{aligned}$$

This notation is convenient for printing, but is not so convenient for actual simplification and working with rational numbers. The students can however do a number of problems in this notation.

Experiment 330. Rational Numbers in Decimal Notation.

A rational number is the ratio of two integers p, q where $q\neq 0$. We consider the case of positive rational numbers only; negative rational numbers differ in sign only.

Let the children try to find possible decimal representation for $\frac{p}{q}$. Since $\frac{p}{q}=(p)\times\left(\frac{1}{q}\right)$, let them start with decimal representation for $\frac{1}{q}$.

$$\begin{aligned}
&\tfrac{1}{2}=\cdot 5,\ \tfrac{1}{3}=\cdot\dot{3},\ \tfrac{1}{4}=\cdot 25,\ \tfrac{1}{5}=\cdot 2,\ \tfrac{1}{6}=\cdot 1\dot{6},\ \tfrac{1}{7}=\cdot\dot{1}4285\dot{7} \\
&\tfrac{1}{8}=\cdot 125,\ \tfrac{1}{9}=\cdot\dot{1},\ \tfrac{1}{10}=\cdot 1,\ \tfrac{1}{11}=\cdot\dot{0}\dot{9},\ \tfrac{1}{12}=\cdot 08\dot{3} \\
&\tfrac{1}{13}=\cdot\dot{0}7692\dot{3}.
\end{aligned}$$

They find decimals which either terminate or show recurrence. Will this always be the case?

When an integer p is divided by another integer q, the possible remainders are $0, 1, 2, 3, \ldots, q-1$. At any stage if the remainder is 0, the procedure terminates and we get a terminating decimal. If the children do not get 0, some remainder will reapeat itself, since we can have only $q-2$ different remainders. As soon as a remainder repeats itself, the recurrence will start.

How do we know which rational numbers give terminating decimals and which give recurring decimals?

Let the children consider a terminating decimal, say

$$\begin{aligned}
&b_m\ldots b_2b_1.a_1a_2\ldots a_n \\
&=b_m\times 10^{m-1}+b_{m-1}\times 10^{m-2}\ldots b_2\times 10^1+b_1+\frac{a_1}{10}+\ldots+\frac{a_n}{10^n} \\
&=\frac{\text{some integer}}{10^n}=\frac{\text{some integer}}{2^n\times 5^n}.
\end{aligned}$$

The numerator may contain some powers of 2 and 5. However, we find that in the rational number representation of a terminating decimal number, the denominator must contain powers of 2 and 5 only *i.e.*, it must be of the form $2^a\,5^b$.

Conversely, consider any rational number of the form $\frac{p}{2^a5^b}$.

Now $\frac{p}{2^a5^b}=\frac{p\times 2^b\times 5^a}{10^{a+b}}=$ a terminating decimal.

Thus the children will see that every terminating decimal can be expressed as a rational number and when expressed in the lowest terms will have only powers of 2 and 5 in the denominator. Conversely, every rational number with only powers of 2 and 5 in the denominator can be expressed as a terminating decimal.

If the denominator consists of powers of other prime numbers like 3, 7, 11, 13, ..., the rational number cannot give rise to a terminating decimal, but it must give rise to a recurring decimal and the recurring period must be less than $q-1$. Conversely, the children can see that every recurring decimal can be expressed as a rational number. We have

$$\tfrac{1}{9}=\cdot 111111\ldots\ldots$$
$$\tfrac{1}{99}=\cdot 0101010101\ldots\ldots$$
$$\tfrac{1}{999}=\cdot 001001001001001\ldots\ldots$$
$$\tfrac{1}{9999}=\cdot 0001000100010001\ldots\ldots$$

We can use these results to express every recurring decimal as a rational number, *e.g.*,

$$\cdot\dot{3}2\dot{5}=\cdot 325\ 325\ 325\ldots\quad=\tfrac{325}{999}$$
$$\cdot\dot{2}45\dot{6}=\cdot 2456\ 2456\ 2456\ldots=\tfrac{2456}{9999}$$
$$\begin{aligned}3\cdot 23\dot{3}4\dot{5}=3\cdot 23345\ 345\ldots\quad&=3\cdot 23+\tfrac{1}{100}\,(\cdot 345345\ldots)\\&=3\tfrac{23}{100}+\tfrac{1}{100}\times\tfrac{345}{999}\\&=\frac{323\times 999+345}{99900}\\&=\frac{323345-323}{99900}\end{aligned}$$

From these and similar examples, the children can deduce the following rule for writing recurring decimal numbers as rational numbers :

In the numerator, write the natural number corresponding to recurring and non-recurring digits and subtract from this the natural number represented by non-recurring digits. In the denominator, write as many nines as there are recurring digits and as many zeros as there are non-recurring digits in the decimal part.

According to this rule,

$$\cdot\dot{9}=\tfrac{9}{9}=1$$

$$1{\cdot}4\dot{9}=\frac{149-14}{90}=1{\cdot}5$$

$$1{\cdot}7\dot{9}=\frac{179-17}{90}=1{\cdot}8.$$

Thus every terminating decimal can also be expressed as a recurring decimal. Thus 2 can be written as $1{\cdot}\dot{9}$, 2·345 can be written as $2{\cdot}344\dot{9}$ and so on. In this sense the expression for rational numbers with powers of 2 and 5 alone in the denominator is not unique, but we can introduce uniqueness by considering recurring decimals only.

Experiment 331. The Set of Terminating Decimals.

This is a sub-set of the set of rational numbers. What is its structure ? Is it a field for the operations of addition and multiplication ?

Let the children verify that this set is closed for addition, the associative law for addition holds, there is an additive identity *viz.* 0, every member has an additive inverse and the commutative law for addition holds. They can also verify that it is closed for multiplication, associative law for multiplication holds, there is a multiplicative identity *viz.* 1, the commutative law for multiplication holds and multiplication distributes addition both on the right and the left. The only law for a field which breaks down is that every non-zero element does not have a multiplicative inverse, *e.g.*, 3·0 has a multiplicative inverse $\frac{1}{3}$ but this cannot be represented as a terminating decimal and as such it is not in our set.

Our present set is however a commutative ring with multiplicative identity.

The order structure is defined here, it follows the trichotomy and transitive laws and is consistent with addition and multiplication structures.

What about denseness ? Many rational numbers (in fact all those which contain any prime numbers other than 2 and 5 in the denominator) are not included in our set. Does it weaken the denseness property ?

Let the children take any two members of this set, say 1·1 and 1·2 and find members of this set between these, *e.g.*,

1·11, 1·12, 1·13, 1·14, 1·15, 1·16, 1·17, 1·18, 1·19 all lie between 1·1 and 1·2.

Are there any more members of this set between 1·1 and 1·2 ? We find 1·111, 1·112, 1·113, 1·114, 1·115, 1·116, 1·117, 1·118, 1·119 lie between 1·11 and 1·12 and so lie between 1·1 and 1·2.

1·1111, 1·1112, 1·1113,......1·1119 lie between 1·111 and 1·112 and so also lie between 1·1 and 1·2.

This process can be continued indefinitely and so between any two numbers represented by terminating decimals, there lies an infinity of numbers of the same type. The property of denseness holds in this sub-set also.

Alternately, what is the number next to any given number say 1·234578932 ? A number after this is 1·23457893200000000001, but we can get still nearer number by choosing more zeros. Thus there is no number of the set which can be regarded as next to the given number.

The children may like to investigate the structures in the sets of recurring decimals, which cannot be expressed as terminating decimals. They would be disappointed as even additive and multiplicative identities are not there. Let them investigate whether denseness is preserved.

4

The Structures of The Real Number System

Experiments : 332 to 342

Experiment 332. Irrational Numbers in Decimal Notation.

Let the children consider 'numbers' like the following :

$a=11{\cdot}01001000100001......$
$b=135{\cdot}2002000020000002......$
$c=246{\cdot}03000300000300000003......$
$d=357{\cdot}11411114111111114......$
....

In the first case, the number of zeros form the sequence 1,2,3,4,...
In the second case, the number of zeros form the sequence 0,2,4,6,...
In the third case, the number of zeros form the sequence 1,3,5,7,...
In the fourth case, the number of ones form the sequence 2,4,8,16,...

Are these rational numbers ? These are neither terminating nor recurring and as such these are not rational numbers. Such numbers whose decimal representation is neither terminating nor recurring are called irrational numbers.

Let the children give a large number of examples of such irrational numbers. They can use any sequence of zeros, ones, twos, threes, ..., nines or even combination of these, *e.g.*,

$f=0{\cdot}11422225111142222251111114......$

Even in the first case instead of using sequence 1,2,3,4,... they can use sequence 2,3,4,5,..., or 3,4,5,6,... or 4,5,6,... and get an infinity of irrational numbers.

Though all these are neither terminating nor recurring, these numbers show a certain pattern and we can say what the millionth or billionth digit would be without actually writing all the digits. In fact, it would provide many interesting problems for the students to find 10,000th digit or 1000,000th digit for each of the numbers they write. However, there are irrational numbers which do not show any such pattern, *e.g.*, consider the number $\sqrt{2}$ which corresponds to the length of diagonal of a square with unit side. In experiment 376, it is shown that $\sqrt{2}$ is not a rational number, so it cannot be expressed as a terminating or recurring decimal. We have a definite algorithm for finding square root, and using it we find

$$\sqrt{2}=1{\cdot}41412\ldots\ldots$$

If we have sufficient patience and time and a fast electronic computer, we can find in principle the millionth digit, but the computer will have to find all the earlier digits also, since there is no pattern in the digits. In principle, we can find any digit we like, though for billionth or trillionth digit, the time required may be more than the life history of the human race. Let the children find the time required for calculating 10^{30}th digit when for each decimal place the time required is 10^{-3} seconds. For 10^{n}th digit, the time required would be of the order 10^{n-10} years. Thus for 10^{30}th digit, the time would be of the order of 10^{20} years.

In principle, however, we can solve such problems. Consider however the following problems. "Does the sequence 123456789 occur anywhere in the decimal expression for $\sqrt{2}$?" We may not be able to answer the question in any finite amount of time, however large.

There are other irrational numbers of the same type, *e.g.*, $\sqrt{3}$, $\sqrt{5}$, $\sqrt[3]{3}$, ... In fact it can be shown (vide experiment 376) that if N and D are not perfect Rth powers, $\sqrt[R]{N/D}$ is an irrational number. The students have so far encountered the following two types of irrational numbers :—

(*i*) Those of the type given in the beginning of this section whose decimal representation shows a certain pattern.

(*ii*) Those of the type $\sqrt[R]{N/D}$ where N, D and R (>1) are natural numbers and N and D are not perfect Rth powers.

But as we shall see, these two classes do not exhaust all irrational numbers.

Experiment 333. Algebraic and Transcendental Numbers.

Consider the number

$$x = \sqrt[R]{\frac{N}{D}}$$

giving $x^R = \frac{N}{D}$ or $D\,x^R = N$.

This equation is called an algebraic equation of the Rth degree.

We may consider more general equations like

$$a_0\,x^n + a_1\,x^{n-1} + \ldots\ldots + a_n = 0$$

where $a_0, a_1, a_2, \ldots\ldots, a_n$ are integers. The elements of truth sets of such algebraic equations are called *algebraic numbers*. Thus $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, …… $\sqrt[3]{3}$, $\sqrt[3]{7}$, …… and in general all numbers of the form $\sqrt[R]{N/D}$ are irrational algebraic numbers. In addition, numbers like $\sqrt{3}+\sqrt{2}$, $\sqrt[3]{5}-\sqrt[3]{2}$ are also algebraic irrational numbers.

Can there be irrational numbers which are not algebraic ?

Two famous numbers which are irrational but not algebraic are π and e. π is defined as the ratio of the circumference of a circle to its diameter. Its approximate values are $\frac{22}{7}$, $\frac{355}{113}$, 3·14159, but it has been proved that it is an irrational number and it cannot satisfy any algebraic equation. We can calculate its value from series like

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\ldots$$

to any number of decimal places. In fact its value has been calculated to more than 10,000 places by using a computer.

e is defined by the series

$$e = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3.} + \frac{1}{1.2.3.4.} + \ldots\ldots$$

For e it has also been shown that it is irrational number but is not an algebraic number. Its value has also been calculated to thousands of places of decimals.

Irrational numbers which are not algebraic are called *transcendental numbers*. Thus e and π are both transcendental numbers.

Experiment 334. Dense-pecking of Real Numbers.

We assume that to every real number there is a unique point on the number line and to every point on the number line, there is a unique real number. Then given any two real numbers a and b and the corresponding points P and Q, we have $b>a$ if Q is on the right of P, $b<a$ if Q is on the left of P and $b=a$ if Q and P coincide.

Given any two positive real numbers,

$$A = a_m \ldots a_3\, a_2\, a_1 . b_1 b_2 \ldots b_p \ldots$$
$$A' = a'_m \ldots a'_3 a'_2 a'_1 . b'_1 b'_2 \ldots b'_p \ldots$$

how do we decide which is greater ? If the number is a terminating decimal, we reduce it to a recurring decimal so that there is a unique representation for every number.

If $m>m'$ then $A>A'$.

If $m=m'$ and $a_m> a'_m$, then $A>A'$.

If $m=m'$, $a_m=a'_m$ and $a_{m-1}>a'_{m-1}$, then $A>A'$.

In general we can say that number is greater whose integral part is greater. If the two integral parts are the same, we compare the decimal digits and find the first decimal digit which is different. The number which has the greater digit is greater. If all the decimal digits are also the same, the two numbers are equal.

Give the children some real numbers and let them find which are greater. If one is negative and the other is positive, the positive number is greater. If both are negative, then the number whose magnitude is smaller is greater.

Is the set of real numbers densely pecked, *i.e.*, given any two distinct real numbers, can we find an infinity of real numbers between them ?

Let A and A′ be distinct positive real numbers and let $A>A'$. Since these are distinct, they must differ in at least one place. Let b_p be the first digit which is greater than b'_p. To get a number between A and A′, if $b_p-b_p'> 1$, we increase b'_p by 1. If $b_p-b'_p=1$, we increase b'_{p+1} by 1 unless $b'_{p+1}=9$ when we replace b'_{p+1} by 0 and increase b'_p by 1 except when $b_{p+1}=0$. In this case we repeat the same procedure with b'_{p+2} except when $b'_{p+2}=1$ and $b_{p+2}=0$. All b's after a stage cannot be zero for in that case all numbers will have a terminating decimal representation, but we have reduced all such numbers to recurring decimals. Thus

Between ·52345...... and ·39874...... we have ·49874......
Between ·42345...... and ·38874...... we have ·39874......
Between ·42345...... and ·39974...... we have ·40974......
Between ·4000001...... and ·3999999...... we have ·4000000......

Thus between any two distinct real numbers, there must be a real number and if there is one, there must be infinitely many.

Give the children many pairs of real numbers and ask them to find real numbers between them.

Thus we find that the set of real numbers is densely pecked. Comparing the system of rational numbers and real numbers, we find

(*i*) The set of rational numbers has a field structure for the operations of addition and multiplication. It can be shown that the set of real numbers also has a field structure for the same operations. We have not however proved this, we have not even defined these operations for real numbers so far.

(*ii*) Both systems have an order structure which is consistent with addition and multiplication structure. Both sets are densely pecked.

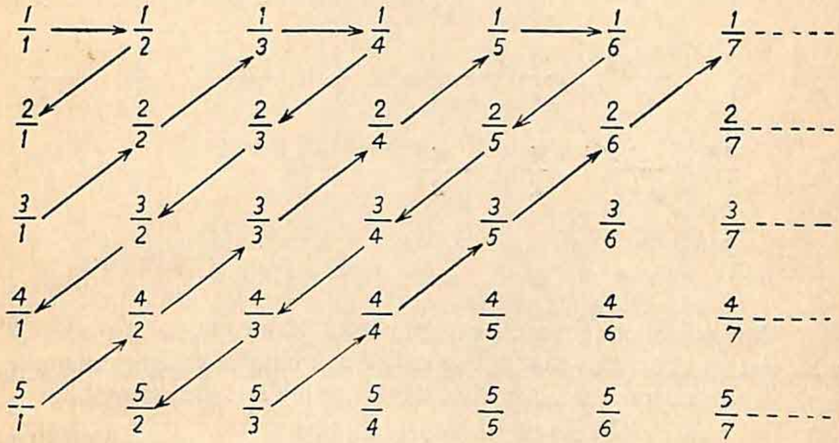
Is there any difference in the two systems ? We proceed to investigate this in the next two experiments.

Experiment 335. Countability of the Set of Rational Numbers.

Between any two rational numbers, there are only a finite number of integers, while there is an infinity of rational numbers. Does it mean that there are more rational numbers than integers ? For answering this question, we have to be clear by what is meant by 'more'.

When we establish a correspondence between two finite sets A and B, we say that A has more members than B if we can establish a correspondence such that to every member of B there corresponds a member of A, but to every member of A there does not correspond a member of B. If we can establish a one-one correspondence between elements of A and B, they have the same number of elements. We extend these concepts now to infinite sets.

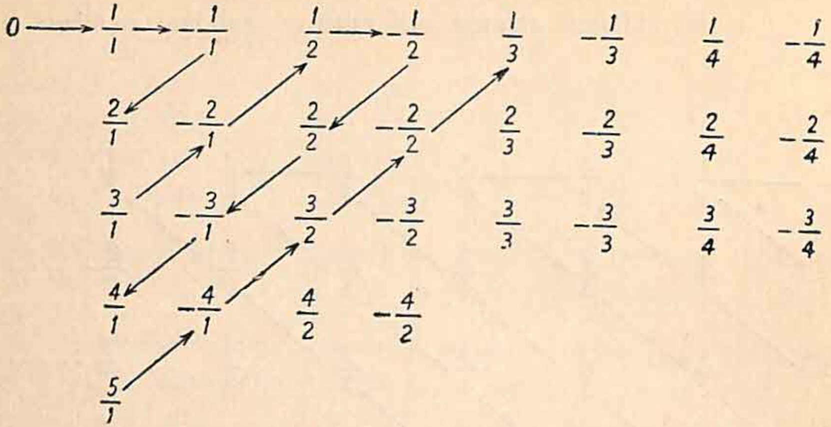
Let the children arrange all positive rational numbers as follows :



In the first row, we have all rational numbers with numerator 1 ; in the second row we have all rational numbers with numerator 2 ; and so on. We establish the following correspondence by following the arrows in the above figures and omitting the numbers which repeat.

Natural Numbers	Rational Numbers
1	$\frac{1}{1}$
2	$\frac{1}{2}$
3	$\frac{2}{1}$
4	$\frac{3}{1}$
5	$\frac{1}{3}$
...	...
...	...

To every natural number, there corresponds a unique positive rational number and to every positive rational number there corresponds a unique natural number. A one-to-one correspondence is established and thus there are 'as many' positive rational numbers as natural numbers. We can also establish a one-one correspondence between elements of Q and N by using the following arrangement :



Any set whose elements can be placed in one-one correspondence with the set of natural numbers is called a countable or an enumerable set. The students can show that the following sets are countable :

(*i*) The set of all natural numbers.
(*ii*) The set of all integers.
(*iii*) The set of all even natural numbers.
(*iv*) The set of all odd natural numbers.
(*v*) The set of all multiples of 3.
(*vi*) The set of all negative rational numbers.
(*vii*) The set of all rational numbers with numerator 5.
(*viii*) The set of all rational numbers between 0 and 1.

Students can suggest other countable sets. All these sets are countable, but all these are not densely pecked. Which ones are densely pecked ?

Experiment 336. Non-countability of Real Numbers.

Since both the systems of rational and real numbers are densely pecked and since the set of rational numbers is countable, the children may suspect that the set of real numbers is also countable. Let us consider the set of real numbers between 0 and 1 and if possible, let there be the following one-one correspondence between the set of natural numbers and the set of real numbers between 0 and 1.

$$\begin{aligned}
1 &\leftrightarrow 0\,.\,a_{11}\,a_{12}\,a_{13}\,a_{14}\,\dots\,\dots\\
2 &\leftrightarrow 0\,.\,a_{21}\,a_{22}\,a_{23}\,a_{24}\,\dots\,\dots\\
3 &\leftrightarrow 0\,.\,a_{31}\,a_{32}\,a_{33}\,a_{34}\,\dots\,\dots\\
\dots\ &\dots\ \dots\ \dots\ \dots\ \dots\,\dots
\end{aligned}$$

If the set is countable, every real number between 0 and 1 should be included somewhere in this correspondence. We now construct a new number as follows :

$0 . b_1 b_2 b_3 b_4 \ldots\ldots$

where b_1 is a digit different from a_{11}
b_2 is a digit different from a_{22}
b_3 is a digit different from a_{33}

and so on.

This is a real number between 0 and 1 and different from the first number in the first decimal place, from the second number in the second place and so on. Thus this is a different number from all those in the list. Thus no countable set can include all real numbers between 0 and 1 and so *ipso facto* it cannot inclue all real numbers.

Thus to every natural number there corresponds a real number (the natural number itself) but to every real number, there does not correspond a natural number.

Thus countability gives one important difference between rational and real number sets. Both sets are infinite, but one set is as ' large' as the set N, other is larger.

What can the children say about the countability of irrational numbers ?

Experiment 337. Dedikand's Approach to Real Numbers.

We have found one difference between rational and real number systems. Dedikand's approach gives another.

Consider the set of all *rational* numbers and divide it into two classes, a lower L and an upper U as follows :

(*i*) Every rational number belongs to L or to U.
(*ii*) Each number of L is less than every number of U.
(*iii*) Neither class is empty.

Dedikand stated that every such classification determines a real number, *e.g.*,

(*a*) Let L consist of all rational numbers $\leqslant 2{\cdot}54$ and U consist of all rational numbers $> 2{\cdot}54$. This determines the real rational number 2·54.

(*b*) Let L consist of all rational numbers $< -3{\cdot}15$ and U consist of all rational numbers $\geqslant -3{\cdot}15$. This determines the real number $-3{\cdot}15$.

(*c*) Let L consist of all negative rational numbers, 0 and all positive rational numbers whose squares are < 2 and let U consist of all positive rational numbers whose squares are > 2. Of course, there is no rational number whose square is equal to 2. This determines the real irrational number $\sqrt{2}$.

In every case, the children can verify that the above three conditions (*i*)—(*iii*) are satisfied.

In cases (*a*) and (*b*), the lower class has a greatest member or the upper class has a least member. In case (*c*), neither L has a greatest member nor U has a least member.

This provides a distinction between rational and irrational numbers. If Dedikand's classification of all rational numbers is such that either the lower class has a greatest member or the upper class has a least member, then the real number determined is a rational number. If neither the lower class has a greatest member nor the upper class has a least member, the real number determined is irrational.

Experiment 338. Nests of Intervals.

We know $\sqrt{2}=1{\cdot}41412......$

Consider the intervals with rational end points :

$(1{\cdot}4, 1{\cdot}5), (1{\cdot}41, 1{\cdot}42), (1{\cdot}414, 1{\cdot}415), (1{\cdot}4141, 1{\cdot}4142), ...$

The method of formulation of these intervals is obvious and the sequence of intervals is an infinite sequence.

We find that

(*i*) Each succeeding interval lies within the preceding.

(*ii*) The lengths of the intervals are $0{\cdot}1, 0{\cdot}001, 0{\cdot}0001, 0{\cdot}00001$, and the lengths become smaller and smaller and can be made as small as we please. We say that the length of the interval approaches zero and that we have a 'nest of intervals'.

It is obvious that $\sqrt{2}$ lies within each of these intervals. Can there be a rational number which lies within each of these intervals ? If there is one, say N, it will differ from $\sqrt{2}$ (since $\sqrt{2}$ is not a rational number). Let $n+1$th place be the first decimal place where it differs from $\sqrt{2}$, *i.e.*, let

$$\sqrt{2}=1.a_1\,a_2\,a_3\ldots a_n\,a_{n+1}\,a_{n+2}\ldots\ldots$$
$$N\ =1.a_1\,a_2\ldots a_n\,b_{n+1}\,b_{n+2}\ldots\ldots$$

We have the following intervals in our nest of intervals,

$$(1.a_1\,a_2\ldots a_{n-1}\,a_n,\ 1.\,a_1\,a_2\ldots a_n\,a_{n+1}),$$
$$(1.a_1\,a_2\ldots a_n a_{n+1},\ 1.a_1\,a_2\ldots a_{n+1+1}),$$
$$(1.a_1\,a_2\ldots a_{n+1}\,a_{n+2},\ 1.a_1\,a_2\ldots a_{n+2+1}).$$

If $b_{n+1}=a_{n+1+1}$, N does not lie in the second interval.

If $b_{n+1}\neq a_{n+1+1}$, N does not lie in the third interval.

There are some special cases when some of the digits are equal to 9 but in every case, it can be shown that no rational number can be in all these intervals. If we consider only rational numbers, then this nest of intervals does not have a common point.

In general, consider the decimal expression

$$N.q_1\,q_2\,q_3\,q_4\,\ldots\ldots$$

which may terminate or recur or not. We find a series of intervals.

$$N.q_1 \leqslant x \leqslant N.\,(q_1+1)$$
$$N.q_1q_2 \leqslant x \leqslant N.q_1(q_2+1)$$
$$N.q_1q_2q_3 \leqslant x \leqslant N.q_1q_2(q_3+1)$$
$$\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots$$

If any of the q's is 9, we can easily modify, *e.g.*, if $q_3=9$, we shall choose the intervals

$$N.q_1q_2q_3 \leqslant x \leqslant N.q_1\,(q_2+1)\,0.$$

Clearly, we have nest of intervals and the given decimal belongs to all of them. Now suppose we consider only rational numbers in the intervals, then if the given number is rational, the nest of intervals has a common member and if the given number is irrational, the nest of intervals has no common member. Thus if we consider rational numbers only and consider a nest of intervals as above, then sometimes the nest will have a common member and sometimes it will not have. On the other hand if we consider all real numbers, then the nest will always have a common member. This provides another and very important difference between rational and real numbers.

Experiment 339. The Usual Algorithm for Finding Square Roots.

The usual algorithm for finding the square root of a number is based on the following identities :—

$$(a+b)^2 = a^2+(2a+b)b$$
$$(a+b+c)^2 = (a+b)^2+\{2(a+b)+c\}c$$
$$(a+b+c+d)^2=(a+b+c)^2+\{2(a+b+c)+d\}d$$

We write these as below. At every stage we double the partial quotient and add a suitable number and multiply the sum by that number.

$$\begin{array}{r|l}
 & a+b+c+d \\ \hline
a & a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd \\
 & a^2 \\ \cline{2-2}
(2a+b) & 2ab+b^2 \\
 & 2ab+b^2 \\ \cline{2-2}
2(a+b)+c & 2ac+2bc+c^2 \\
 & 2ac+2bc+c^2 \\ \cline{2-2}
2(a+b+c)+d & 2ad+2bd+2cd+d^2 \\
 & 2ad+2bd+2cd+d^2 \\ \cline{2-2}
 & \times
\end{array}$$

In numerical problems, we also use the place value system so that the successive numbers in the result give successive decimal places.

Let the students understand clearly the reasoning behind the usual algorithm for finding the square root of a number and find square root of some numbers by explicitly using the identities. Let them try to extend it for finding cube roots, fifth roots etc.

Experiment 340. Another Algorithm for Finding Square Roots.

Another algorithm is obtained by considering the relation

$$x_{n+1}=\frac{1}{2}\left(x_n+\frac{A}{x_n}\right)\ (n=2,\ 3,\ 4,\ \ldots)$$

where A is the positive number whose square root is to be found. We give any value we like to x_1, then this determines x_2 and then substituting for x_2, we get x_3 and the process can be continued till either the numbers repeat themselves or till we find that the difference in succeeding numbers is less than the accuracy required, *e.g.*, let us use it to find the square roots of 4 and 2. In each case, we choose $x_1=1$.

(*a*) **Square root of 4**

$$x_1=1,\ x_2=\frac{1}{2}\left(1+\frac{4}{1}\right)=2\cdot 5,\ x_3=\frac{1}{2}\left(2\cdot 5+\frac{4}{2\cdot 5}\right)=2\cdot 05$$

$$x_4=\frac{1}{2}\left(2\cdot 05+\frac{4}{2\cdot 05}\right)=2\cdot 00061.$$

We find that the values are coming very rapidly nearer and nearer to 2 but will never reach 2 in a finite amount of time, but since in practice we shall need the value of square root correct to a certain number of places, we can stop the process after a certain stage.

Suppose we had, on the other hand chosen $x_1=2$, then

$$x_1=2,\ x_2=\frac{1}{2}\left(2+\frac{4}{2}\right)=2,\ x_3=2,\ x_4=2,\ \ldots\ldots$$

and then we get the exact answer.

(*b*) **Square root of 2**

$$x_1=1,\ x_2=\frac{1}{2}\left(1+\frac{2}{1}\right)=\frac{3}{2},\ x_3=\frac{1}{2}\left(\frac{3}{2}+\frac{4}{3}\right)=\frac{17}{12},$$

$$x_4=\frac{1}{2}\left(\frac{17}{12}+\frac{24}{17}\right)=\frac{577}{408},\ \ldots\ldots$$

We shall find that the values are coming nearer and nearer to $\sqrt{2}$.

How are we sure that the above relation will always give values which will be coming closer and closer to $\sqrt{A}$?

$$x_{n+1}-\sqrt{A}=\frac{1}{2}\left(x_n+\frac{A}{x_n}-2\sqrt{A}\right)=\frac{1}{2}\left(\sqrt{x_n}-\frac{\sqrt{A}}{\sqrt{x_n}}\right)^2$$

$$=\frac{1}{2x_n}(x_n-\sqrt{A})^2$$

Similarly,

$$x_n-\sqrt{A}=\frac{1}{2x_{n-1}}(x_{n-1}-\sqrt{A})^2$$

$$x_2-\sqrt{A}=\frac{1}{2x_1}(x_1-\sqrt{A})^2$$

We easily see that

(*i*) $x_n-\sqrt{A}>0$ for $n>1$.

(*ii*) If $x_1\neq\sqrt{A}$, then none of $x_2, x_3 \ldots x_n \ldots$ is equal to $\sqrt{A}$.

(*iii*) If $x_1=\sqrt{A}$, then all of $x_2, x_3 \ldots x_n, \ldots$ are equal to $\sqrt{A}$.

(*iv*) Since

$$\frac{x_{n+1}-\sqrt{A}}{x_n-\sqrt{A}}=\frac{1}{2}\left(1-\frac{\sqrt{A}}{x_n}\right)<\frac{1}{2},$$

difference between each successive approximate and $\sqrt{A}$ is less than half the difference between the preceding approximation and $\sqrt{A}$ so that there is rapid approach to $\sqrt{A}$. In fact nearer x_n gets to $\sqrt{A}$, smaller $\frac{1}{2}\left(1-\frac{\sqrt{A}}{x_n}\right)$ becomes and better becomes the next approximation.

The algorithm simply states that to get the square root of A, choose an arbitrary positive number, take the average of this number and the number obtained by dividing A by this number to get a new number. Again take the average of this new number and the number obtained by dividing A by this new number to get another number. Now proceed with this number as before.

We are assured by the above arguments that the above process will give us the value of $\sqrt{A}$ correct to as many places as we like. In fact it will give much quicker convergence than the usual algorithm. However, it is not a very convenient process since it involves division, but even then the total labour involved is not more. With a slide rule, a calculating machine or a computer, this is more convenient.

Experiment 341. Other Algorithms for Finding Square Roots, Cube Roots, etc.

(*i*) For finding square roots, the children can use the algorithm based on the relation

$$(1+k)\,x_{n+1}=k\,x_n+\frac{A}{x_n},$$

where A is the positive number whose square root is to be found and k is any fixed number different from -1. As special cases, we have

$$k=1,\quad x_{n+1}=\frac{1}{2}\left(x_n+\frac{A}{x_n}\right).$$

$$k=2,\quad x_{n+1}=\frac{1}{3}\left(2x_n+\frac{A}{x_n}\right).$$

$$k=3,\quad x_{n+1}=\frac{1}{4}\left(3x_n+\frac{A}{x_n}\right).$$

In each case, we can start with any suitable value of x_1 (usually we choose the positive integer just less than $\sqrt{A}$) and calculate the sequence $\{x_1, x_2, x_3, x_4, \ldots\}$. Each of these sequences will converge to $\sqrt{A}$. It will be interesting to investigate how the convergence depends on k.

(*ii*) For finding cube roots, the children can use the algorithm based on the relation

$$x_{n+1}=\frac{1}{2}\left(x_n+\frac{A}{x_n^2}\right)$$

or the more general relation

$$(1+k)\,x_{n+1}=k\,x_n+\frac{A}{x_n^2},$$

where A is the positive number whose cube root is to be found and k is any fixed number different from -1.

(*iii*) Similarly for finding the mth root, the children can use the algorithm based on the relation

$$x_{n+1}=\frac{1}{2}\left(x_n+\frac{A}{x_n^{m-1}}\right),$$

or the more general relation

$$(1+k)x_{n+1}=k\,x_n+\frac{A}{x_n^{m-1}}.$$

The children can also investigate how the convergence depends on k and m.

(*iv*) Another algorithm for finding the square root of 2 is based on 'side and diameter' numbers, attributed to Pythagoras. The algorithm is based on the relations

$$x_{n+1}=x_n+2y_n,\ y_{n+1}=x_n+y_n$$

so that

$$x_{n+1}^2-2y_{n+1}^2=-(x_n^2-2y_n^2).$$

If we put $x_1=1$, $y_1=1$, we get

$$x_n^2-2y_n^2=(-1)^n$$

and

$$\frac{x_n^2}{y_n^2}-2=\frac{(-1)^n}{y_n^2},$$

so that the sequence $\frac{x_n}{y_n}$ rapidly converges to $\sqrt{2}$.

The successive values we get for the approximation are :

$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{119}, \ldots\ldots$

Experiment 342. Properties of Number Systems.

After having studied the properties of the various number systems, let the children be encouraged to prepare the following table :

Property	*Natural Numbers*	*Whole Numbers*	*Integers*	*Positive Rationals*	*Rational Numbers*	*Real Numbers*
(*a*) **Addition**	N	W	I	Q_+	Q	R
(*i*) Closure	✓	✓	✓	✓	✓	✓
(*ii*) Commutative law	✓	✓	✓	✓	✓	✓
(*iii*) Associative law	✓	✓	✓	✓	✓	✓
(*iv*) Existence of additive identity (0)	×	✓	✓	×	✓	✓
(*v*) Existence of additive inverse for every element	×	×	✓	×	✓	✓
(*vi*) Subtraction always possible	×	×	✓	×	✓	✓
(*b*) **Multiplication**						
(*i*) Closure	✓	✓	✓	✓	✓	✓
(*ii*) Commutative law	✓	✓	✓	✓	✓	✓
(*iii*) Associative law	✓	✓	✓	✓	✓	✓
(*iv*) Existence of multiplicative identity	✓	✓	✓	✓	✓	✓
(*v*) Existence of multiplicative inverse for every element	×	×	×	✓	✓ (except for 0)	✓ (except for 0)
(*vi*) Division always possible	×	×	×	✓	✓ (except for 0)	✓ (except for 0)
(*c*) **Addition-Multiplication**						
(*i*) Multiplication distributes addition on the right	✓	✓	✓	✓	✓	✓
(*ii*) Multiplication distributes addition on the left	✓	✓	✓	✓	✓	✓
(*d*) **Other Properties**						
(*i*) Order structure	✓	✓	✓	✓	✓	✓
(*ii*) Densely pecked	×	×	×	✓	✓	✓
(*iii*) Existence of smallest element	✓	×	×	×	×	×
(*iv*) Nesting property	×	×	×	×	×	✓
(*v*) Countably infinite	✓	✓	✓	✓	✓	×

The students can also extend the table for finite modular systems and for complex numbers.

5

Binary Operations

Experiments : 343 to 349

Experiment 343. Laws for Binary Operations.

Consider the set of all natural numbers. If whenever we combine two natural numbers in some specified way, we always get a natural number, then this way of combining is called a binary operation in the set of natural numbers. Thus addition and multiplication are binary operations in N, since the sum and product of two natural numbers is always a natural number. Subtraction and division are not binary operations in N as the result of subtracting or dividing one natural number by another natural number is not always a natural number. In the set I of integers, subtraction is also a binary operation, since the result of subtracting an integer from another integer is always an integer. In the set of non-zero rational numbers, division is a binary operation, since the result of dividing one non-zero rational number by another non-zero rational number is always a non-zero rational number.

Let a, b be any two arbitrary elements of a set and let $a \, o \, b$ denote the result of combining these members so that $a \, o \, b$ is also a member of the set, then the symbol o denotes a binary operation. This operation is called **commutative** if for all a, b in the set

$$a \, o \, b = b \, o \, a$$

This operation is called associative, if for all a, b, c in the set

$$(a \, o \, b) \, o \, c = a \, o \, (b \, o \, c)$$

If there are two binary operations * and o defined on the set, then * *is said to distribute* o, if for all a, b, c in the set

$$a * (b \, o \, c) = (a * b) \, o \, (a * c)$$
$$(b \, o \, c) * a = (b * a) \, o \, (c * a)$$

The operation o is said to possess an *identity element* e if there exists an element e of the set such that for all members a of the set

$$a \; o \; e = e \; o \; a = a$$

Further if an operation has an identity element e, then two elements a and b of the set are said to be *inverses* of each other if

$$a \; o \; b = b \; o \; a = e$$

We now illustrate these concepts :

(*i*) Consider the set N. Let $a \; o \; b$ denote a^b. If a and b are natural numbers, a^b is also a natural number and so this operation is a binary operation. It is not commutative since for all a, b in N

$$a^b \neq b^a,$$

It is not associative since for all a, b, c in N,

$$(a^b)^c \neq a^{(b^c)}$$

It does not possess an identity element, since while

$$a^1 = a, \; 1^a \neq a.$$

(*ii*) In N, let $a \; o \; b$ denote $a+2b$, so that, *e.g.*,

$$3 \; o \; 4 = 11, \; 4 \; o \; 3 = 10$$
$$3 \; o \; (4 \; o \; 5) = 3 \; o \; (14) = 31,$$
$$(3 \; o \; 4) \; o \; 5 = 11 \; o \; 5 = 21$$

Thus the commutative and associative laws do not hold, nor is there an identity element.

(*iii*) In N, let $a \; o \; b$ denote $a \times b^2$ so that, *e.g.*,

$$3 \; o \; 4 = 48, \; 4 \; o \; 3 = 36$$
$$(3 \; o \; 4) \; o \; 5 = 48 \; o \; 5 = 1200, \; 3 o \; (4 o 5) = 3 o \; (100) = 30{,}000$$

Again the commutative and associative laws do not hold, nor is there an identity element.

(*iv*) In N, let $a \; o \; b$ denote $a+b+ab$ so that, *e.g.*,

$$\begin{aligned} &3 \; o \; 4 = 19, && 4 \; o \; 3 = 19, \\ &(3 \; o \; 4) \; o \; 5 = 19 \; o \; 5 && = 119, \\ &3 \; o \; (4 \; o \; 5) && = 3 \; o \; (29) = 119 \\ &a \; o \; b = a+b+ab, && b \; o \; a = b+a+ba \\ &(a \; o \; b) \; o \; c = (a+b+ab) \; o \; c && = a+b+ab+c+(a+b+ab)c \\ & && = a+b+c+ab+ac+bc+abc \\ &a \; o \; (b \; o \; c) = a \; o \; (b+c+bc) && = a+b+c+bc+a(b+c+bc) \\ & && = a+b+c+ab+ac+bc+abc \end{aligned}$$

Both the commutative and associative laws hold, but the operation does not possess an identity element.

(*v*) In I, let $a * b$ denote $a+b+ab$, then the operation is both commutative and associative and possesses an identity element *viz*., since

$$a*0 = a+0+a \, . \, 0 = a$$
$$0*a = 0+a+0 \, . \, a = a$$

Every element however does not have an inverse for if b is the inverse of a, then

$$a+b+ab=0, \; b=-\frac{a}{1+a}$$

If a is an integer, b need not be an integer. However, 0 and -2 are inverses of themselves.

(*vi*) In N, let $a \circ b$ denote the H.C.F. of a and b, then $a \circ b = b \circ a$ since the H.C.F. of a and b is the same as that of b and a and $(a \circ b) \circ c = a \circ (b \circ c) =$ H.C.F. of the three numbers a, b, c. Thus the binary operation of taking H.C.F. of two natural numbers is both commutative and associative, but does not possess an identity element.

(*vii*) If $a \circ b$ denotes L.C.M. of a and b then this operation is commutative, associative, has an identity element 1 and relatively prime natural numbers are inverses of each other, so that each number has an infinite number of inverse elements.

(*viii*) In N, let $a \circ b$ denote a^2+b^2 then it is commutative, but is not associative and does not possess an identity element.

(*ix*) In N, the following are binary operations :
$a \, f \, b = a + 2b$, $a \, f \, b = a^b$, $a \, f \, b = a^{b^2}$, $a \, f \, b = a \times b^2$
$a \, f \, b = a \times 10^b$, $a \, f \, b = a^2 b^3$, $a \, f \, b = a + 2^b$

For each of these, commutativity, associativity, existence of identity element and existence of inverse elements can be examined.

(*x*) For any pair of operations, we can discuss whether either distributes the other, *e.g.*, in N, let
$a \, f \, b = a \times b$, $a \, g \, b = a^b$, then
$a \, f \, (b \, g \, c) = a \times b^c$, $(a \, f \, b) \, g \, (a \, f \, c) = (ab) g \, (ac) = (ab)^{ac}$
and f does not distribute g.

(*xi*) In N, I and Q, multiplication distributes addition. In I and Q, multiplication distributes subtraction. In Q_0, *i.e.*, in the set of non-zero rational numbers, addition and subtraction are distributed on the right by division, since if a, b, c are in Q_0, then

$$\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c},\quad \frac{a-b}{c}=\frac{a}{c}-\frac{b}{c}.$$

The students may be encouraged to define other binary operations N, I, Q and examine their commutativity, associativity, existence of an identity, existence of inverse elements and distributivity of one operation over another.

Experiment 344. Commutative Law.

In the systems of natural numbers, we have already seen that the additive and multiplicative structures are commutative. For subtraction, if $a-b$ is a natural number, $b-a$ will not even be a natural number, the question of their equality does not arise. For division, both $a \div b$ and $b \div a$ may not be natural numbers and it is possible that one is a natural number then the other will not be unless $a=b$.

Not only in mathematics, but also in ordinary life, sometimes the change of order is permitted, but very often it is not, *e.g.*, consider the following examples :

(*i*) Put on your socks, then put on your shoes.
Put on your shoes and then put on your socks.

(*ii*) Put on your banyan, then put on your shirt.
Put on your shirt and then put on your banyan.

(*iii*) Put the milk in cup and then add tea to it.
Put tea in the cup and then add milk to it.

(*iv*) Take off your clothes and then take a bath.
Take bath first and then take off your clothes.

(*v*) Add water to sulphuric acid.
Add sulphuric acid to water.

(*vi*) Take one step forward and then turn left.
Turn left and then take one step forward.

(*vii*) Take one step towards the North and then one step towards the East.
Take one step towards the East and then one step towards the North.

(*viii*) Strip the door and then paint it.
Paint the door and then strip it.

(*ix*) First print the pages of a book and then bind these.
First bind the pages together and then print them.

(*x*) First let the enemy attack and then prepare for defence.
First prepare for defence and then let the enemy attack.

(*xi*) First teach and then examine.
First examine and then teach.

Any number of such examples can be constructed. These examples show that it is really fortunate that commutative law holds for both addition and multiplication. Our arithmetic is very much simplified because of these laws. Subtraction and division are not commutative, but fortunately their inverse operations of addition and multiplication are. Mathematicians also study non-commutative operations because these are necessary for practical applications, but their study is more complicated than that of commutative operations. Later the students will study entities called vectors, matrices, operators etc. for which certain operations are not commutative.

Experiment 345. Associative Law.

The student may consider whether the following are true :

(*i*) (Tea+milk)+sugar=Tea+(milk+sugar)
i.e. whether we add milk to tea first and then add sugar to this mixture or we add to tea a mixture of milk and sugar, will the result be the same ?

(*ii*) (Sugar+eggs)+flour=Sugar+(eggs+flour)
On the left, we add eggs to sugar and then add flour to the mixture of eggs and flour, but on the right we first add eggs to the flour and then add sugar to the mixture of eggs and flour. The result in both the cases is the same.

(*iii*) (Typing+printing)+binding=Typing+(printing+binding)
In the first case, we may do typing and printing first and then give the result for binding. In the second case, we type first and then give the manuscript for printing and binding.

It is because of the associative law that we can use the shorter notation a^4 for writing $a \times a \times a \times a$. If $(a \times a) \times a$ was different from $a \times (a \times a)$, we could not have used the same notation a^3 for both. Even our place value system which is so useful is based on this law. If $(300+40)+5$ was different from $300+(40+5)$, we could not have used the notation 345 for both.

It will therefore be useful, if pointed attention is drawn to commutative and associative laws whenever they are used in arithmetic and algebra.

Experiment 346. Distributive Laws.

There are four operations, *viz.*, addition, subtraction, multiplication and division which we have considered so far and each can possibly distribute the other giving rise to 12 possible laws. Further the distribution can be either on the right or on the left and thus the total number of possible distributive laws becomes 24. We enumerate these below :

(*i*) $a\times(b+c)=(a\times b)+(a\times c)$, (*ii*) $(b+c)\times a=(b\times a)+(c\times a)$

(*iii*) $a\times(b-c)=(a\times b)-(a\times c)$, (*iv*) $(b-c)\times a=(b\times a)-(c\times a)$

(*v*) $a\times(b\div c)=(a\times b)\div(a\times c)$, (*vi*) $(b\div c)\times a=(b\times a)\div(c\times a)$

(*vii*) $a+(b\times c)=(a+b)\times(a+c)$, (*viii*) $(b\times c)+a=(b+a)\times(c+a)$

(*ix*) $a+(b\div c)=(a+b)\div(a+c)$, (*x*) $(b\div c)+a=(b+a)\div(c+a)$

(*xi*) $a+(b-c)=(a+b)-(a+c)$, (*xii*) $(b-c)+a=(b+a)-(c+a)$

(*xiii*) $a\div(b+c)=(a\div b)+(a\div c)$, (*xiv*) $(b+c)\div a=(b\div a)+(c\div a)$

(*xv*) $a\div(b-c)=(a\div b)-(a\div c)$, (*xvi*) $(b-c)\div a=(b\div a)-(c\div a)$

(*xvii*) $a\div(b\times c)=(a\div b)\times(a\div c)$, (*xviii*) $(b\times c)\div a=(b\div c)\times(c\div a)$

(*xix*) $a-(b+c)=(a-b)+(a-c)$, (*xx*) $(b+c)-a=(b-a)+(c-a)$

(*xxi*) $a-(b\times c)=(a-b)\times(a-c)$, (*xxii*) $(b\times c)-a=(b-a)\times(c-a)$

(*xxiii*) $a-(b\div c)=(a-b)\div(a-c)$, (*xxiv*) $(b\div c)-a=(b-a)\div(c-a)$

These laws will be said to hold in a system if they hold for *all* members of the system. It is possible that an equality may not hold for all members of a system, but may hold for some special members. Thus for the system of integers I (*i*) to (*iv*) hold for all members, (*v*) and (*vi*) hold if $a=1$, (*vii*) to (*xii*) hold if $a=0$, (*xiv*) holds if $b\div a$ and $c\div a$ are integers. The students may similarly discuss all the laws for systems N, I and Q (the system of rational numbers).

For other systems and other operations, some of these other distributive laws may hold, *e.g.*, the students will later find that for the system of sub-sets of a given set and for the operations of union and intersection, each operation will distribute the other.

The distributive laws

$$a\times(b+c)=a\times b+a\times c$$

and

$$(b+c)\div a=b\div a+c\div a \quad (a\neq 0)$$

are fundamental for the operations of multiplication and division of numbers.

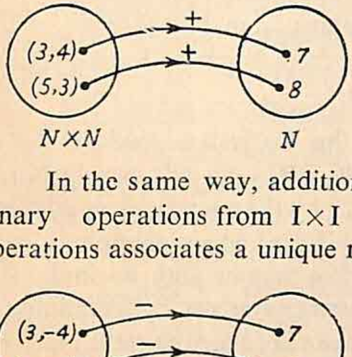
Experiment 347. Binary Operations as Mappings.

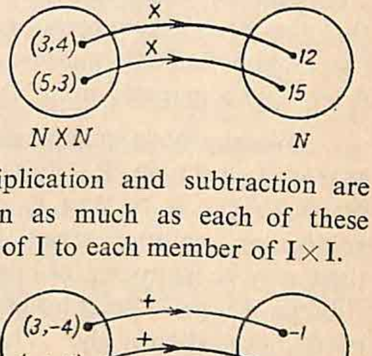
A mapping from a set X to a set Y is a rule which associates to every member of X a unique member of Y.

The Cartesian product of two sets X and Y is the set of all ordered pairs (x, y) where x is an element of X and y is an element of Y.

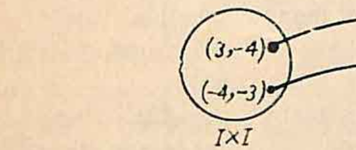
Binary Operations can be regarded as mappings.

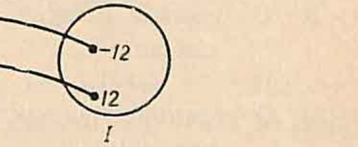
Thus for the set N of natural numbers, the binary operations of addition and multiplication can be regarded as mappings from N×N to N in as much as to every ordered pair of natural numbers, the binary operations of addition associates a unique natural number and the binary operation of multiplication associates another unique natural number.

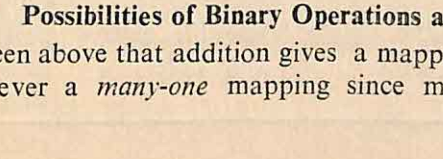




In the same way, addition, multiplication and subtraction are binary operations from I×I to I in as much as each of these operations associates a unique member of I to each member of I×I.



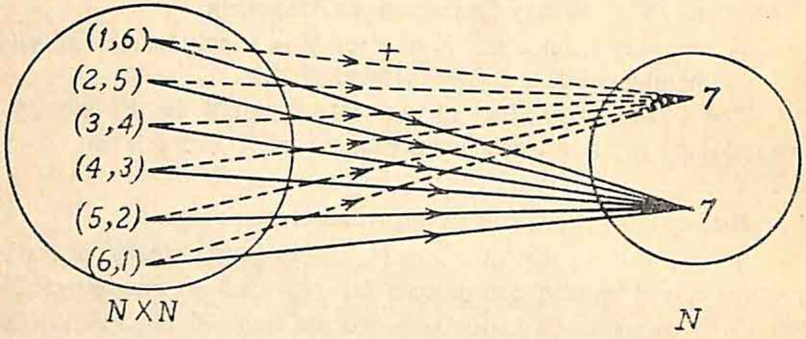




Experiment 348. Possibilities of Binary Operations as Mappings.

We have seen above that addition gives a mapping from N×N to N. It is however a *many-one* mapping since many members of

$N \times N$ are associated with the same member of N. Thus the following mapping is many-one.



It is also an into mapping, since there is no member of $N \times N$ which is associated with the member 1 of N.

Also subtraction and division are not binary operations in N and thus they cannot be represented as mapping from $N \times N$ to N. However, subtraction determines a mapping from $N \times N$ to I and division determines a mapping from $N \times N$ to Q.

We may have mappings from the Cartesian products $N \times N$, $W \times W$, $I \times I$, $Q \times Q$, $R \times R$, $Q_0 \times Q_0$, $R_0 \times R_0$, $N \times W$, $N \times I$, $N \times Q$, $N \times R$, $I \times Q$,...to N, W, I, Q, Q_0, R, R_0, by the operations of addition, subtraction, multiplication, division. Even when mappings exist, these may be many-one or one-one, these may be onto or into. The children will get a great insight into number systems by examining all possibilities exhaustively. Thus we have the following table :—

From	To	+	×	−	÷
$N \times N$	N	possible, many-one, into	possible, many-one, onto	not possible	not possible.
$N \times N$	Q	possible, many-one, into	possible, many-one, into	possible many-one, onto	not possible.
$Q \times Q$	Q	possible, many-one, onto	possible, many-one, onto	possible, many-one, onto	not possible.
$Q \times Q_0$	Q	possible, many-one, onto	possible, many-one, onto	possible, many-one, onto	possible, many-one, onto

Let the children examine all the possibilities and prepare their own tables. They can also include sets with finite number p of elements and addition and multiplication modulo p, where p may or may not be prime.

Experiment 349. Inverse Mappings for Binary Operations.

Let
$$f : X \to Y$$
be a mapping from a set X to a set Y so that to every member x of X we can associate a unique member y of Y. We shall say that an inverse mapping f^{-1} exists or
$$f^{-1} : Y \to X$$
if there exists a rule by which to every y of Y we can associate same unique x of X to which y of Y corresponds by f.

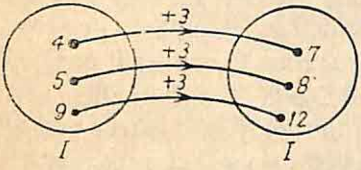
Thus if f is the mapping from R which associates to a real number its double, then f^{-1} is the inverse mapping also from R to R which associates to every real number its half.

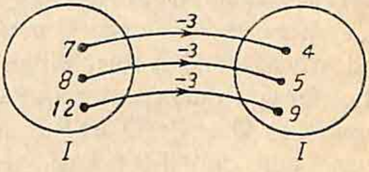
We usually say that addition and subtraction are inverse operations and so are multiplication and division. Can we say that inverse operations give rise to inverse mappings ?

Thus though addition and subtraction both provide mappings from $I \times I$, but we cannot say that the mappings are inverse, for while addition is a mapping from $I \times I$ to I, subtraction is not a mapping from I to $I \times I$.

Can we consider these as inverse mappings in any other sense ?

We consider the mappings $(+3)$ and (-3) from I to I, defined below :





In this case the two mappings are inverse of each other. Similarly, $(\times 3)$ and $(\div 3)$ can be regarded as giving inverse mappings from Q_0 to Q_0.

The mapping $(\times \frac{2}{3}) : N \to Q_0$ exists.
The mapping $(\div \frac{2}{3}) : Q_0 \to N$ does not exist.

6

Equivalent Sentences

Experiments : 350 to 355

Experiment 350. Equivalent Equations and Inequations.

An open sentence is a statement containing one or more variables like x, y, z etc. or containing frames like □, △, ▢ etc. In their places, we can write numbers from a certain *substitution set* or a *replacement set*, *e.g.*, from the set of natural numbers or of integers or of rational numbers or of real numbers. For some substitutions, the sentences become true statements. These values constitute the truth set of the open sentences.

An open sentence whose verb is $=$ is called an *equation* while an open sentence whose verb is $<$ or $>$ or $\leqslant$ or $\geqslant$ is called an *inequality* or an *inequation*.

Sentences which have the same truth set are called *equivalent sentences*. One of the important problems in mathematics is that of finding sentences equivalent to given sentences in as 'simple'a form as possible. Two sentences P and Q are said to be equivalent if they have the same truth set. In this case we write $P \equiv Q$ and note that whenever P is true, Q must be true (*i.e.*, P implies Q or $P \Rightarrow Q$) and whenever Q is true, P must be true (*i.e.*, Q implies P or $Q \Rightarrow P$). Thus for equivalent sentences $P \Rightarrow Q$ and $Q \Rightarrow P$, *i.e.*, P implies and is implied by Q or $P \Leftrightarrow Q$ or P is true if and only if Q is true and Q is true if and only if P is true. $\Rightarrow$ is called one-way implication and $\Leftrightarrow$ or $\equiv$ is called a two-way implication.

Experiment 351. Rules for Finding Equivalent Equations.

When the substitution set is the set of real numbers, equations equivalent to a given equation can be obtained by using *one or more* of the following rules :

(*i*) If we add the same number to both sides, we get an equivalent equation, *e.g.*,
$x+5=2x+3 \equiv x+5+6=2x+3+6.$

(*ii*) If we subtract the same number from both sides, we get an equivalent equation, *e.g.*,
$x+5=2x+3 \equiv x+5-6=2x+3-6.$

(*iii*) If we multiply both sides of an equation by the same number ($\neq 0$), we get an equivalent equation, *e.g.*,
$x+5=2x+3 \equiv 2\,(x+5)=2(2x+3).$

(*iv*) If we divide both sides of an equation by the same number ($\neq 0$), we get an equivalent equation, *e.g.*,
$x+5=2x+3 \equiv \frac{1}{2}\,(x+5) = \frac{1}{2}\,(2x+3).$

We may note that the operations of squaring and taking absolute values do not give equivalent equations. Thus

$x=5 \not\equiv x^2=25$, since $x=5 \Rightarrow x^2=25$ while $x^2=25 \not\Rightarrow x=5$ only.

Again

$x=-5 \not\equiv |x|=5$, since $x=-5 \Rightarrow |x|=5$ while $|x|=5 \not\Rightarrow x=-5.$

Experiment 352. Justification of the Use of the Adjective 'Equivalent'.

Every equation is a sentence and two sentences are said to be equivalent, if each implies the other, *i.e.*, whenever one is true, the other is also true, so that the truth set of one coincides with that of other. It is easily seen that each of the above four operations leads to equivalent sentences. This follows from the laws of structures of the real number system which state that

$$a=b \Leftrightarrow a+c=b+c$$
$$a=b \Leftrightarrow a-c=b-c$$
$$a=b \Leftrightarrow ac=bc \qquad (c\neq 0)$$
$$a=b \Leftrightarrow \frac{a}{c}=\frac{b}{c} \qquad (c\neq 0)$$

Can the students understand the reasoning ?

Experiment 353. Equivalence Relation.

If two equations are equivalent, there is a relation between them. This relation is an equivalence relation, for

(*i*) The relation is *reflexive*, since every equation is equivalent to itself as it can be obtained from itself by adding or subtracting 0 from both sides or by multiplying both sides by unity.

(*ii*) The relation is *symmetric*, since if one equation is equivalent to another, then the second is equivalent to the first. Thus if the second equation is obtained by adding (or multiplying by) some number to both sides of the first equation, the first equation can be obtained from the second equation by subtracting (or dividing by) the same number from both sides of the second equation. This symmetry is made possible since every real number has an additive inverse and every non-zero real number has a multiplicative inverse.

(*iii*) The relation is *transitive*, since if the first equation is equivalent to the second and the second equation is equivalent to the third, then the first is equivalent to the third. Thus if the second equation is obtained from the first by addition of a number a to both sides and the third is obtained from the second by multiplying both sides by the number b, then the third is obtained from the first by the compound operation of first adding a to both sides and then multiplying both sides by b.

Thus the relation of equivalence between equations is an equivalence relation and partitions all equations into disjoint classes, such that all equations within a class are equivalent to one another and no equation of one class is equivalent to an equation of another class. How many equations belong to an equivalence class?

Our object is to get the simplest equation of a class which is of the form $x=\ldots$, $y=\ldots$, etc.

Experiment 354. Examples of Equivalent Equations.

(*i*)

$$\begin{aligned} & x+1=0 \\ \Leftrightarrow\ & (x+1)-1=0-1 \\ \Leftrightarrow\ & x+(1-1)=-1 \\ \Leftrightarrow\ & x+0=-1 \\ \Leftrightarrow\ & x=-1 \end{aligned}$$

(*ii*)

$$\begin{aligned} & x+1=2x+4 \\ \Leftrightarrow\ & 2x+4=x+1 \\ \Leftrightarrow\ & (2x+4)-x=(x+1)-x \\ \Leftrightarrow\ & x+4=1 \\ \Leftrightarrow\ & (x+4)-4=1-4 \\ \Leftrightarrow\ & x=-3 \end{aligned}$$

(*iii*)*

$$\begin{array}{llll} & x+y & =2 & \wedge\ x-y =0 \\ \Leftrightarrow & (x+y)+(x-y) & =2+0 & \wedge\ (x+y)-(x-y)=2-0 \\ \Leftrightarrow & 2x & =2 & \wedge\ 2y =2 \\ \Leftrightarrow & x & =1 & \wedge\ y =1. \end{array}$$

Experiment 355. Equivalent Inequations.

Remembering the laws for consistency of order structure with addition and multiplication structures of real numbers, we can obtain equivalent inequalities by using one or more of the following operations :

(*i*) Adding or subtracting the same number to both sides

$x>y \Leftrightarrow x\pm z>y\pm z$

$x<y \Leftrightarrow x\pm z<y\pm z$

$x\geqslant y \Leftrightarrow x\pm z\geqslant y\pm z$

$x\leqslant y \Leftrightarrow x\pm z\leqslant y\pm z.$

(*ii*) Multiplying or dividing both sides by the same positive number

$x>y\Leftrightarrow xz>yz,\quad x>y\Leftrightarrow \dfrac{x}{z}>\dfrac{y}{z} \qquad (z>0)$

$x<y\Leftrightarrow xz<yz,\quad x<y\Leftrightarrow \dfrac{x}{z}<\dfrac{y}{z} \qquad (z>0)$

$x>y\Leftrightarrow xz>yz,\quad x\geqslant y\Leftrightarrow \dfrac{x}{z}\geqslant\dfrac{y}{z} \qquad (z>0)$

$x\leqslant y\Leftrightarrow xz\leqslant yz,\quad x\leqslant y\Leftrightarrow \dfrac{x}{z}\leqslant\dfrac{y}{z} \qquad (z>0).$

(*iii*) Multiplying or dividing both sides by the same negative number reverses the inequality sign,

$x>y\Leftrightarrow xz<yz,\quad x>y\Leftrightarrow \dfrac{x}{z}<\dfrac{y}{z} \qquad (z<0)$

$x<y\Leftrightarrow xz>yz,\quad x<y\Leftrightarrow \dfrac{x}{z}>\dfrac{y}{z} \qquad (z<0)$

$x\geqslant y\Leftrightarrow xz\leqslant yz,\quad x\geqslant y\Leftrightarrow \dfrac{x}{z}\leqslant\dfrac{y}{z} \qquad (z<0)$

$x\leqslant y\Leftrightarrow xz\geqslant yz,\quad x\leqslant y\Leftrightarrow \dfrac{x}{z}\geqslant\dfrac{y}{z} \qquad (z<0).$

---

* The symbol $\wedge$ stands for 'and'. P $\wedge$ Q is true when both P and Q are true and is false when either of them is false.

It can be shown by result that the relation of equivalence in equalities is an equivalence relation.

We give below two examples of equivalent inequations :—

(*i*) $x+3>2x+4$

$\Leftrightarrow \quad 3>x+4$

$\Leftrightarrow x+4<3$

$\Leftrightarrow \quad x<-1$

(*ii*) $\frac{x}{2}+y+6>2x+4y+5$

$\Leftrightarrow \quad 6>\frac{3}{2}x+3y+5$

$\Leftrightarrow \frac{3}{2}x+3y+5<6$

$\Leftrightarrow \quad \frac{3}{2}x+3y<1$

$\Leftrightarrow \quad 3x+6y<2.$

7

Systems of Numeration

Experiments : 356 to 363

Experiment 356. The Hindu-Arabic System of Numeration.

The present system of numeration that we use all over the world, was evolved in India and was passed on to Europe by the Arabs. The important characteristics of this system are (*i*) existence of 0 and (*ii*) place-value system. These are two important gifts of India to the world. Without these, modern civilization would have been almost impossible. Our students, as students all over the world, must realise the importance of this system. For this purpose, let them consider some of the other systems used in ancient times.

Experiment 357. The Greek System of Numeration.

The Greeks used the following symbols :

1	α	alpha	10	ι	iota	100	ρ	rho
2	β	beta	20	κ	kappa	200	σ	sigma
3	γ	gamma	30	λ	lambda	300	τ	tau
4	δ	delta	40	μ	mew	400	υ	upsilon
5	ε	epsilon	50	ν	new	500	φ	phi
6	ς	stigma	60	ξ	axi	600	χ	chi
7	ζ	zeta	70	ο	omicron	700	ψ	psi
8	η	eta	80	π	pi	800	ω	omega
9	θ	theta	90	ϰ	kappa	900	ϡ	saipi

In our system, we use only ten digits *viz.*, 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 and can represent all numbers, however large. In the Greek system, they used twenty-seven symbols and yet they could not represent large numbers. They had no place-value system and no zero. The Greeks were good in geometry, but in arithmetic and algebra, they could not go as far as the Indians because of this cumbersome system of numeration.

The students can write some numbers in the Hindu-Arabic system in the Greek system and conversely. Thus 8675 would be written as $\omega\omega\omega\omega\omega\omega\omega\omega$ % $o\ \epsilon$ and $\omega\ \psi\ \pi\ \delta$ will stand for 1584.

Some practice with this system is desirable for another reason. In all higher mathematics and science, the Greek alphabet is used freely and it is desirable that students become familiar with this alphabet at an early stage. In fact the word 'alphabet' comes from alpha, beta, the first two letters in the Greek language.

The students may also try some addition, multiplication and division sums in this system to realise the tremendous advantage of the present Hindu-Arabic system. Thus

$$\omega\ \chi\ \pi\ \delta + \varrho\ \xi\ \epsilon = \omega\ \chi\ \varrho\ \pi\ \xi\ \delta\ \epsilon = \omega\ \omega\ \mu\ \theta$$

$$\begin{aligned} \omega\ \pi\ \eta\ \chi\ \epsilon &= \omega\ \omega\ \omega\ \omega\ \omega\ \pi\ \pi\ \pi\ \pi\ \pi\ \eta\ \eta\ \eta\ \eta\ \eta \\ &= \omega\ \omega\ \omega\ \omega\ \omega\ \bar{\nu}\ \mu \end{aligned}$$

A multiplication sum like 8456732 × 3214356 would be almost impossible and may take a full day and a full exercise book.

Experiment 358. The Roman System.

The Romans used the symbols

1	I	6	VI	50	L
2	II	7	VII	100	C
3	III	8	VIII	500	D
4	IV	9	IX	1000	M
5	V	10	X		

The system does not have a zero, though it has a partial place value system as indicated in IV, VI, IX, XI, XL. The system is still used for many purposes, though it is not used for adding, multiplication and division.

We can use borrowing, carrying over and distributive laws. Thus

$$\begin{aligned}
\text{MMMMCCCLXIV}-\text{MMCCLXIX} &= \text{MMCXIV}-\text{XIX}\\
&= \text{MMCV}-\text{X}=\text{MMLXXXXV}\\
(\text{CXLIV})\times(\text{LV}) &= (\text{C}+\text{X}+\text{L}+\text{V}-\text{I})(\text{L}+\text{V})\\
&= \text{C}\times\text{L}+\text{X}\times\text{L}+\text{L}\times\text{L}+\text{V}\times\text{L}-\text{I}\times\text{L}\\
&\quad +\text{C}\times\text{V}+\text{X}\times\text{V}+\text{L}\times\text{V}+\text{V}\times\text{V}-\text{I}\times\text{V}\\
&= \text{MMMMM}+\text{D}+\text{MMD}+\text{CCL}-\text{L}\\
&\quad +\text{D}+\text{L}+\text{CCL}+\text{XXV}-\text{V}\\
&= \text{MMMMMMMDDDCCCCLXX}\\
\text{MDCLVI}\div\text{VI} &= \text{M}\div\text{VI}+\text{D}\div\text{VI}+\text{C}\div\text{VI}+\text{L}\div\text{VI}\\
&\quad +\text{VI}\div\text{VI}\\
&= \text{MCC}\div\text{VI}+\text{CCCCXX}\div\text{VI}+\text{XXXVI}\div\text{VI}\\
&= \text{CC}+\text{LXX}+\text{VI}=\text{CCLXXVI}.
\end{aligned}$$

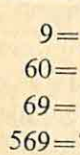
Experiment 359. The Egyptian System.

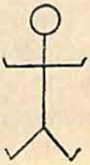
The Egyptians used the following symbols :

1 1

10 ∩

100 ?

1000 

1,000,000 

Thus in this system

9=111111111

60=∩∩∩∩∩∩

69=∩∩∩∩∩∩111111111

569=? ? ? ? ? ∩∩∩∩∩∩111111111

There was no zero, no place value system and large numbers were difficult to represent. Addition was performed in the obvious way, *e.g.*,

∩∩111+∩∩11=∩∩∩∩11111

but multiplication was done by successive doubling and addition. Thus to multiply by 19, they would multiply the number successively four times by 2, then once by 2, then by 1 and then add the three numbers.

Thus

∩∩1111×∩1=(∩∩1111×11×11×11)+(∩∩1111×11)
+(∩∩1111×1)
=(∩∩∩∩11111111×11×11)
+(∩∩∩∩11111111)+∩∩1111
=(∩∩∩∩∩∩∩∩∩111111×11)
+∩∩∩∩∩∩∩11
=? ∩∩∩∩∩∩∩∩∩11+∩∩∩∩∩∩∩11
=? ? ∩∩∩∩∩∩ 1111

representing 24×11=264.

Experiment 360. Advantages of Hindu-Arabic System.

(*i*) The numerals for some numbers are shorter here than in other systems.
(*ii*) Large numbers can be represented easily.
(*iii*) The four operations can be performed even with very large numbers in much less time and much less space.
(*iv*) The operations are easier to perform.

The advantages are due to the existence of 0 and the place value system. The use of ten as a base is however arbitrary and some of the advantages persist even when we use other bases. Thus with 2 as base, though the numerals become longer and multiplication and division take more time, yet the remarkable fact is that with only two symbols *viz.* 0 and 1, we can represent every number however large. With 12 as base, the symbols increase by 2, but numerals become shorter and calculations also become shorter.

Experiment 361. Distinction between Numbers and Numerals.

What is common between one horse, one cow, one table, one chair, one flower, one tree etc. ? It is the property of 'oneness' or 'being single'. They all illustrate the number '1'. Similarly, the property common to all pairs of objects is 'twoness' and they illustrate the number '2', and the property common to all triplets of objects is 'threeness' and they all illustrate the number '3'.

Thus number is an abstract concept. Thus '5' is the common property of all sets of objects which can be placed in one-to-one correspondence with the set of 5 stars

* * * * *

i.e., we consider all sets which are such that, to every element of each set there corresponds a star and to every star, there corresponds an element of the set. The abstract property which is common to all such sets is the number '5'.

Every abstract concept requires a concrete representation. Numeral is a concrete representation for a number.

A child may have any number of names, yet the child is the same. A number is the same whatever representation we use for it. Thus in different world languages the number five is represented by different symbols or different numerals, but the number five is the same, whatever numerals we use for it. Thus in India, we use the following numerals :—

ENGLISH	:	1	2	3	4	5	6	7	8	9
HINDI	:	१	२	३	४	५	६	७	८	९
URDU	:	۱	۲	۳	۴	۵	۶	۷	۸	۹
BENGALI	:	১	২	৩	৪	৫	৬	৭	৮	৯
GUJARATI	:	૧	૨	૩	૪	૫	૬	૭	૮	૯
MARATHI	:	१	२	३	४	५	६	७	८	९
PUNJABI	:	੧	੨	੩	੪	੫	੬	੭	੮	੯
ORIYA	:	୧	୨	୩	୪	୫	୬	୭	୮	୯
MALYALAM	:	൧	൨	൩	൪	൫	൬	൭	൮	൯
TELGU	:	౧	౨	౩	౪	౫	౬	౭	౮	౯
KANNADA	:	೧	೨	೩	೪	೫	೬	೭	೮	೯

The students may write some numbers say 854, 1239, 50678 using different numerals.

Experiment 362. Different Names for Numbers.

A number can have different names, *e.g.*,

$$8=1+7=6+2=4+4=11-3=100-92=16\div2=64\div8$$
$$=4\times2=2^3=2\times2+4=\cdot4\times20=\sqrt{64}=\ldots\ldots\ldots\ldots$$

These are all different names for the number '8'. Obviously, every number can have an infinity of names.

The students can write as many different names for a given number as they like.

The students may also be asked to give names of numbers in terms of one digit, say 9 only, *e.g.*,

$$1=9\div 9,\ 2=(9\div 9)+(9\div 9),\ 3=(9+9+9)\div 9$$
$$10=9+(9\div 9),\ 101=99+(9+9)\div 9=999\div 9-9-(9\div 9).$$

The students may write names from numbers 1 to 50 in terms of one digit only say 1 or 2 or 3 or 4... or 9, using the digit a minimum number of times.

They may also write names for numbers 1 to 100 in terms of one digit, say 4, using it a fixed number, say 4 times. Thus

$$1=44\div 44,\ 2=(4\div 4)+(4\div 4),\ 3=\sqrt{4}+\sqrt{4}-(4\div 4)$$
$$9=\frac{4}{4}+4+4,\ 24=4\times 4+4+4,\ \ 26=\frac{4}{\cdot 4}+4\times 4,\ \ 44=\frac{4}{\cdot 4}\times 4+4$$
$$100=\frac{4}{\cdot 4}\times\frac{4}{\cdot 4}\cdot$$

A board can be hung in the class room with numbers 1 to 100 on it. A student can write the representation for any number in terms of four four's only and write his or her name against it. The board can continue to hang in the class room unless names have been found for all the numbers.

The students can also write names for all numbers in terms of two digits, using these a number of times. Thus

$$5=4+(3\div 3),\ 8=4+4,\ 15=3\times 4+3$$
$$25=4\times 4+3\times 3,\ 0=3-3,\ -2=3+3-4-4,$$
$$-7=3\times 3-4\times 4,\ \ldots\ldots\ldots\ldots$$

Arithmetic may be regarded as the science of different names for numbers. Thus when we want to multiply 3245 by 3567, we want one number name for 3245×3567.

Experiment 363. An Alternative Way of Bringing Home Distinction between Numbers and Numerals.

The following statements illustrate the point we have been making:

(*i*) '1' and '1' is '11'.

(*ii*) The (right) half of '8' is '3'.

(*iii*) The (upper) half of 'XI' is 'VI'.

(*iv*) 'Smiles' is the largest word in English language because there is a 'mile' between the first and the last letter.

(*v*) Nobody can drink 'water'.

(*vi*) 'Water' cannot put out fire.

(*vii*) You can write 'red' and 'blue' with a black lead pencil.

(*viii*) '**3**' is bigger than '8'.

(*ix*) One-third of 'III' is 'I'.

(*x*) He walked on 'FIRE' and 'WATER' and sat under a 'TRAIN'.

8

Patterns in Numbers, Logic and Sets

Experiments : 364 to 369

Experiment 364. Reversed Addition Process to be Continued Till a Symmetric Number is Obtained.

This is a continuation of experiment 48.

We take a two-digit number, reverse its digits and add the two numbers. We repeat this process with the new number and go on repeating this process till a symmetric number (*i.e.*, a number whose digits equidistant from the left and the right are the same) is obtained. For all two-digit numbers, the process ends in a finite number of steps. The largest number of steps is required by the number 89. Thus we have—

```
 12     23     37     59     69
 21     32     73     95     96
 --     --    ---    ---    ---
 33     55    110    154    165
              011    451    561
              ---    ---    ---
              121    605    726
                     506    627
                    ----   ----
                    1111   1353
                           3531
                           ----
                           4884
```

Let the children prepare a table of the number of steps required for each of the numbers 10 to 99. Let them also continue it for at least some three-digit numbers.

They can also modify the reversed addition process by omitting the leftmost digit whenever they get a three-digit number on adding. The same process can be done for three or four digit numbers and in each case the process will terminate when there is recurrence or when a symmetrical number is obtained. The children can again prepare a table of the number of steps required for each number.

59	75	89	379
95	57	98	973
——	——	——	———
×54	×32	×87	×352
45	23	78	253
——	——	——	———
99	55	×65	605
		56	506
		——	———
		×21	×111
		12	
		——	
		33	

The children can do similar problems for multiplication.

Experiment 365. A Combination of Binary and Denary Systems.

This experiment is a continuation of Experiments 97—100 and 290—292.

We have a number of boxes each with four cells. The boxes may be coloured differently. In each cell, there can be a bead and the value of the bead depends on the cells and the boxes as follows :

IV		III		II		I	
8000	4000	800	400	80	40	8	4
2000	1000	200	100	20	10	2	1

A bead in the four cells of box I represents 1, 2, 4, 8 respectively, a bead in the four cells of box II represents 10, 20, 40, 80 respectively, a bead in the four cells of box III represents 100, 200, 400, 800 respectively and so on. The rules are :

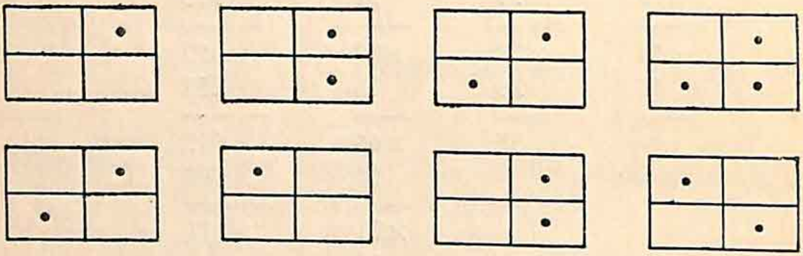
(*i*) No cell is to have more than one bead.

(*ii*) Two beads in a cell of a box are equivalent to one bead in the next cell of the same box.

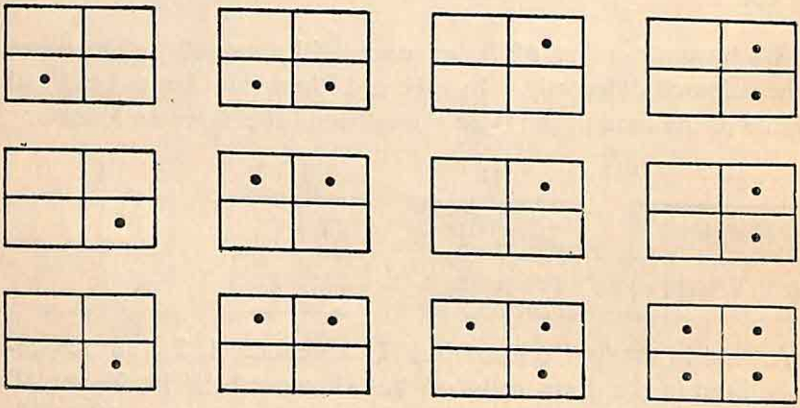
(*iii*) Two beads in the last cell of a box are equivalent to one bead in the first cell of the next box together with one bead in each of second and third cells of the original box.

The rules (*ii*) and (*iii*) are easily seen to be true.

Any number can be represented in this system, *e.g.*, the following two rows represent the numbers 4567 and 6859 respectively.

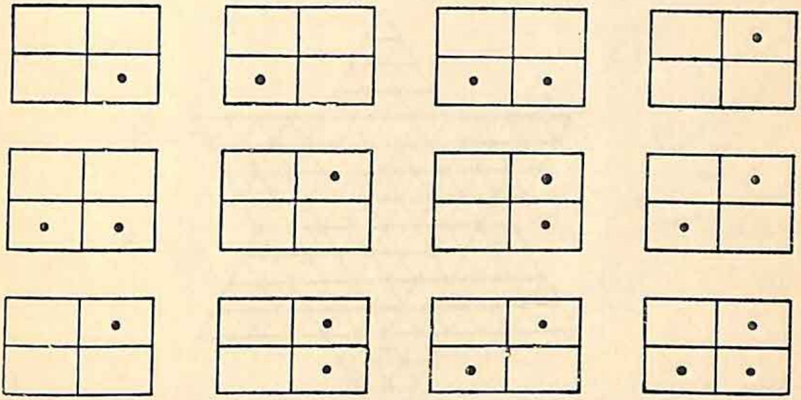


We represent units in the first box, tens in second box, hundreds in the third box, thousands in the fourth box and so on. With this convention, the representation is unique. Without this convention, it is not unique, *e.g.*, each of the following three rows represents the same number, *viz.*, 2345.

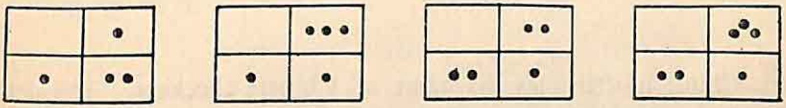


It will be an interesting combinatorial problem to find out the number of possible representations for a given number. However, we shall use the above convention to get a unique representation.

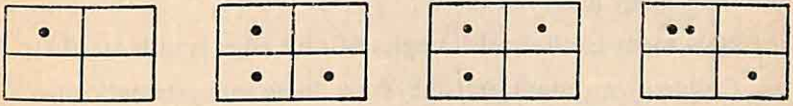
Let us now see how to add two or more numbers with the help of these boxes, *e.g.*, let us find 1234+3456+4567.



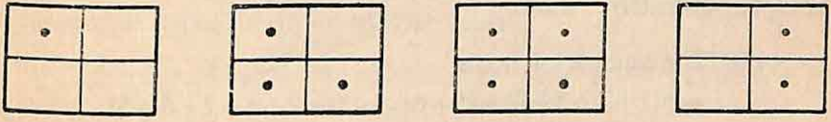
If we pool the beads together, we get—



Now we use rule (*ii*) above to get—



Now we use rule (*iii*) to get—



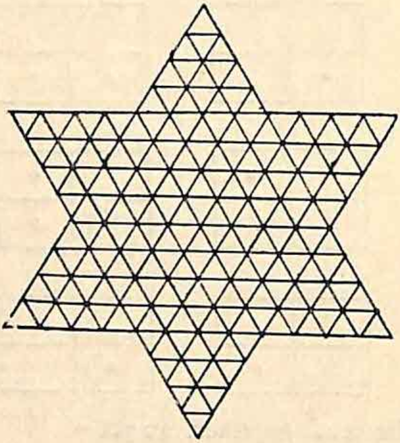
The required sum is therefore

8000+1100+150+7
=9257

With practice, the children can do addition and subtraction problems quite quickly.

Instead of using boxes, the children can use abacuses with four spikes each.

Experiment 366. A Pattern in Numbers.



Children often play the game of Chinese checkers. The design gives rise to many problems of recognition of patterns in numbers.

How many holes are there ?

How many equilateral triangles of unit edge length are there ?

Children can count, but they have to count systematically.

They can count holes in the six corner triangles and the central hexagon separately. Thus—

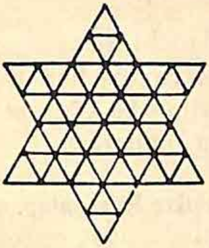
(*i*) The number of holes

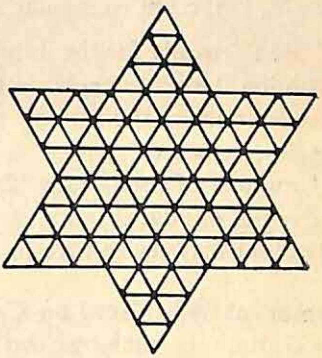
$$
\begin{aligned}
&=6(1+2+3+4)+(5+6+7+8+9+8+7+6+5)\\
&=4(1+2+3+4)+2(1+2+3+4+5+6+7+8)+9\\
&=40+72+9\\
&=121.
\end{aligned}
$$

(*ii*) The number of triangles

$$
\begin{aligned}
&=6(1+3+5+7)+(9+11+13+15+15+13+11+9)\\
&=4(1+3+5+7)+2(1+3+5+7+9+11+13+15)\\
&=64+128\\
&=192.
\end{aligned}
$$

Let the children now draw central hexagons with edge lengths 2, 3, 4, 5, 6, 7, 8,... units and complete the patterns as below :





Let them write their results as follows :

Length of edge of the central hexagon	Number of holes	Number of triangles
2	$6(1+2)+2(3+4)+5 = 37$	$6(1+3)+2(5+7) = 48$
3	$6(1+2+3) + 2(4+5+6)+7=73$	$6(1+3+5)+2(7+9+11) = 108$
4	$6(1+2+3+4) + 2(5+6+7+8)+9=121$	$6(1+3+5+7) + 2(9+11+13+15)=192$

The pattern is now obvious *e.g.* if the length is 100 units, the number of holes $=6(1+2+3+4+\ldots+100)+2(101+\ldots+200)+201$ and the number of triangles$=6(1+3+5+\ldots+199)+2(201+\ldots+399)$. In general, if the length of an edge is n units,

$$\begin{aligned}\text{number of holes} &= 6(1+2+\ldots+n)+2(\overline{n+1}+\ldots+2n)+2n+1\\ &= 4(1+2+\ldots+n)+2(1+2+\ldots+2n)+2n+1\\ &= 4\times\frac{n(n+1)}{2}+2\times\frac{2n\times(2n+1)}{2}+2n+1\\ &= 6n^2+6n+1\end{aligned}$$

and number of triangles

$$\begin{aligned}&= 2(1+3+5+\ldots\text{to } 2n \text{ terms})\\ &\quad +4(1+3+5+\ldots\text{to } n \text{ terms})\\ &= 2(2n)^2+4n^2\\ &= 12n^2.\end{aligned}$$

The formula for the number of holes can be expressed in terms of triangular numbers. Thus number of holes$=2T_{2n}+4T_n+2n+1$, where T_n is the nth triangular number.

The formula for the number of triangles can easily be obtained by folding the six triangles onto the hexagon so that the area of the figure is equal to the sum of areas of 12 triangles and each small triangle consists of $1+3+\ldots+(2n-1)=n^2$ small triangles so that the total number of triangles is $12n^2$. The children may find the number of triangles of edge 2 units, 3 units, 4 units in the Chinese checkers design and may try to investigate the evolving patterns.

Experiment 367. Need for Caution in Inductive Reasoning.

Intuition, making conjectures and inductive reasoning play important roles in mathematics. However, a great deal of care is needed in arriving at results by inductive reasoning. For final certainty, we have to use deductive reasoning. This fact is amply illustrated below :

(*i*) We find

$$\frac{16}{64}=\frac{1}{4},\ \frac{19}{95}=\frac{1}{5},\ \frac{49}{98}=\frac{4}{8},\ \frac{26}{65}=\frac{2}{5}.$$

From these examples, it appears that if we cancel the common digit from the numerator and denominator, we shall get the right result, but this generalisation is not valid, *e.g.*,

$$\frac{15}{52}\neq\frac{1}{2},\ \frac{14}{43}\neq\frac{1}{3}.$$

In fact the above examples are exceptions. Let the students construct more examples of this type with three or four digits in numerator and denominator.

(*ii*) Consider n^2+n+41

When $n=0$, we get 41 which is a prime number.
When $n=1$, we get 43 which is a prime number.
When $n=2$, we get 47 which is a prime number.
When $n=3$, we get 53 which is a prime number.
When $n=4$, we get 61 which is a prime number.
When $n=5$, we get 71 which is a prime number.

Can we generalise that n^2+n+41 will always give a prime number for all natural numbers n ?

We find that this generalisation is not valid since when $n=40$, we get 1681 which is a composite number since it is equal to 41×41.

If we put $n=41$, $n=82$ or 123 or any other multiple of 41, we again get composite numbers.

(*iii*) Another formula that was proposed for giving prime numbers only was $2^{2^n}+1$.

When $n=1$, we get 5 which is a prime number.
When $n=2$, we get 17 which is a prime number.
When $n=3$, we get 257 which is a prime number.
When $n=4$, we get 63537 which is a prime number.
When $n=5$, we get 4,294,967,297 which is however composite, since it has 641 as a factor. What is the other factor? Again we cannot generalise from four cases.

(*iv*) We know 2, 3, 5, 7, 11, 13, ... are prime numbers.

We also know that
$2+1$, $2\times3+1$, $2\times3\times5+1$, $2\times3\times5\times7+1$, $2\times3\times5\times7\times11+1$ are all prime numbers.

We conjecture that if we multiply the first n consecutive prime numbers and add 1, we shall always get a prime number. The students can easily prove this conjecture to be false since the next three numbers of this class are divisible by 59, 19, 347 respectively.

(*v*) We write first 25 natural numbers in 5 rows of 5 numbers each. We find each row contains at least one prime. Next we write first 36 natural numbers in 6 rows of 6 numbers each. We again find each row contains at least one prime number. Next we write first 49 natural numbers in 7 rows of 7 numbers and again find that each row contains at least one prime number. We conjecture that if we write the first n^2 natural number in n rows of n numbers each, each row will contain at least one prime number. This conjecture has been verified or proved for all values of n upto 4500, yet we cannot say that the result is true for all values of n, since no proof for the general result is so far known. Let the children verify the result upto $n=20$.

(*vi*) Consider

$$x^2+y^2=z^2$$

and let the students find positive integers x, y, z which satisfy the equation. There can be an infinite number of Pythagorean triplets like (3, 4, 5), (6, 8, 10), (9, 12, 15), (5, 12, 13), (10, 24, 26) etc. as solutions of the equation.

Consider

$$x^3+y^3=z^3$$

and let the students find positive integers to satisfy the equation. They cannot find the solution in positive integers of $x^3+y^3=z^3$, $x^4+y^4=z^4$, $x^5+y^5=z^5$, ... This leads them to Fermat's last theorem *viz.*, the result that

$$x^n+y^n=z^n$$

has no solution in positive integers for $n>2$. This result has been proved for some values of n but inspite of all efforts it has not been possible to prove it for all n.

(*vii*) 11 is prime.

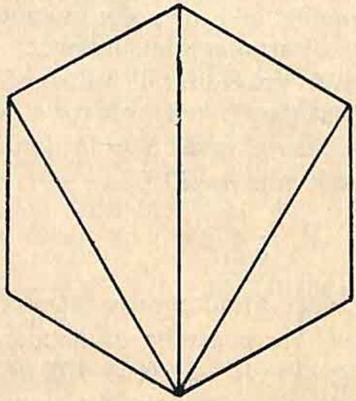
$111=3\times37$; $1111=11\times101$; $11111=41\times271$;
$111{,}111=11\times10101$; $11{,}111{,}111=239\times4649$.

We conjecture that all numbers, except 11, whose digits are one's only are composite. This result holds for numbers upto 22 digits, but the number consisting of 23 digits *viz.*,

11,111,111,111,111,111,111,111

is known to be prime. Let the children check up.

(*viii*) The sum of the angles of a triangle is 2 right angles. The sum of the angles of a quadrilateral is 4 right angles, the sum of the angles of a pentagon is 6 right angles and the sum of the angles of a hexagon is 8 right angles. Let the students generalise and state that the sum of the angles of a polygon of n sides is $2(n-2)$ right angles. This generalisation is correct and can be established by dividing the polygon into $n-2$ triangles.



(*ix*) We find twin primes, *i.e.*, pairs of prime numbers which differ only by 2 like (3, 5), (11, 13), (17, 19) (29, 31), (41, 43) and so on. We find more than 8000 such pairs in numbers less than a million. We conjecture that the number of twin primes is infinite, but the conjecture remains unproved till today.

(*x*) We observe the sequence of numbers 2, 4, 8, ... What are the next three numbers in this sequence ?

  (*a*) We note that each number is double of the preceding. On this basis, the next three numbers in the sequence are : 16, 32, 64.

  (*b*) We note that to get the second number from the first, we add 2 and to get the third number from the second, we add 4. On this basis, we subsequently add 6, 8, 10 to get the three numbers 14, 22, 32.

  (*c*) We note that the formulae n^2-n+2 fits the three numbers. On this basis the next three numbers are 14, 22, 32.

  (*d*) We note that the formulae $\frac{1}{3}(8n-3n^2+n^3)$ also fits the three numbers. On this basis, the next three numbers are 16, 30, 52.

Thus from a knowledge of three numbers, we cannot say what the next numbers are going to be. Suppose the next observation comes to be 16. This would rule out (*b*) and (*c*).

If the next observation comes out to be 30, this would rule out (*a*) also.

The scientist usually gets a few observations, guesses a formula and takes more observations. So long as the formula is satisfied, he continues to accept it. When the formula breaks down, he modifies it to suit new observations. He continues this process till he gets a formula which is always satisfied, but even if the formula is verified 1000 or 10,000 times, it does not prove the formula. The larger the number of verifications, the greater is the confidence of the scientist in his formula. The proof, if any, has to be mathematical.

The mathematician also observes patterns in numbers or space, guesses formulae and then tries to prove these. If he can prove these, he gets theorems, if he gets examples in which the formulae are false, he disproves the formulae while if he can neither prove these, nor find counter-examples to disprove these, the results remain conjectures to be proved or disproved by future mathematicians. From the above discussion, it appears that 'observing of patterns' and 'generalisation' remain the lifeblood of mathematics and science. This power of keen observation of patterns and generalisation should be developed in the first seven or eight years of schools. At the same time the need for proof can be emphasized. After the seventh or eighth year, the capacity for proving has gradually to be developed. The power of mathematics lies in its capacity not only to generalise, but also in its power to prove the generalisations.

The teacher can scan the literature and construct a large number of both valid and invalid generalisations.

Experiment 368. Dependence of the Truth Set of an Open Sentence on the Basic Set or the Universal Set under Consideration.

In most of the problems, it is important to state explicitly the basic set (or the universal set) under consideration, since the solution set or truth set of an open sentence very much depends on the basic set. We illustrate this important fact below :

(*a*) Consider the open sentence $\square + \triangle = 5$.

If the basic set is N, the truth set is $\{(1, 4), (2, 3), (3, 2), (4, 1)\}$.

If the basic set is W, the truth set has additional members $(0, 5)$, $(5, 0)$.

If the basic set is the set of integers 0, 1, 2, 3, 4, 5, 6 and addition is modulo 7, the truth set has additional member $(6, 6)$.

If the basic set is the set of all integers, the truth set has additional members of the form $(6, -1)$, $(7, -2)$, $(8, -3)$, ...

If the basic set is the set of positive rationals, the truth set has also members of the form $(\frac{1}{2}, 4\frac{1}{2})$, $(3\frac{1}{2}, 1\frac{1}{2})$ $(\frac{2}{3}, 4\frac{1}{3})$, ...

If the basic set is the set of all rationals, the truth set has also members of the form $(-\frac{1}{2}, 5\frac{1}{2})$ $(-7, 12)$ $(-3{\cdot}5, 8{\cdot}5)$,......

If the basic set is the set of all real numbers, the truth set has also members of the form $(\pi, 5-\pi)$, $(e, 5-e)$, $(\sqrt{2}, 5-\sqrt{2})$,......

If the basic set is the set of natural numbers greater than 5, then the truth set is the null set.

The general solution consists of all ordered pairs $(x, 5-x)$, where

For N, x is a natural number <5.

For W, x is a whole number $\leqslant 5$.

For I, x is any integer.
For Q, x is any rational number.
For R, x is any real number.

The dependence of the truth set on the basic set under consideration is obvious.

(*b*) Similarly, consider the open sentence $\square + \triangle < 5$.

Its general solution consists of all ordered pairs $(x, r-x)$ where

For N, r is a natural number $\leqslant 4$ and x is a natural number $< r$.
For W, r is a whole number $\leqslant 4$ and x is a whole number $\leqslant r$.
For I, r is any integer $\leqslant 4$ and x is any integer.
For Q, r is any rational number < 5 and x is any rational number.
For R, r is any real number < 5 and x is any real number.

Thus $(\pi, \frac{1}{2}e)$ belongs to the solution set for R, but not for Q.
$(3\cdot5, 1\cdot4)$ belongs to the solution set for Q, but not for I.
$(-10, 14)$ belongs to the solution set for I but not for W.
$(0, 4)$ belongs to solution set for W, but not for N.

(*c*) Consider the open sentence $\square+\triangle=5, 2\square+4\triangle=19$.

Its truth set for N, W and I is ϕ, while for Q or R, the truth set is $\{(\frac{1}{2}, 4\frac{1}{2})\}$.

(*d*) The truth set of $\square+\triangle=5$, $2\square+2\triangle=6$ is ϕ in every case.

(*e*) The truth set of $\square^2-(\sqrt{3}+\sqrt{2})\square+\sqrt{6}=0$ is ϕ for N, W, I and Q and consists of 2 elements, *viz.*, $\sqrt{2}$, $\sqrt{3}$ for R.

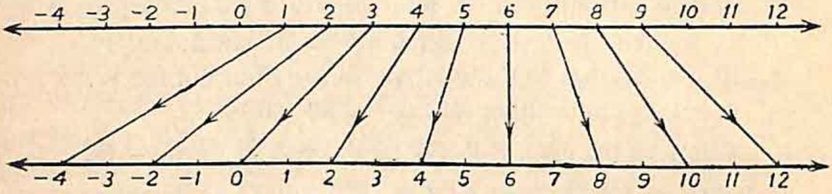
(*f*) The truth set of $ax^2+bx+c=0$ has two members, if the basic set for x is the set of complex numbers. In other cases it may have no member, 1 member or 2 members, *e.g.*, the truth set of $x^2+1=0$ is null if the basic set for x is N or I or Q or R and consists of one member *viz.* 1 if the basic set for x is the set of numbers 0, 1 and addition and multiplication are modulo 2.

(*g*) The fundamental theorem of algebra states that the open sentence $a_0x^n+a_1x^{n-1}+\ldots+a_n=0$ has always n members if the basic set for x is the set of complex numbers and a's belong to N or W or I or Q or R or C except in the case when all the a's are zero in which case it has an infinite truth set. If the basic set for x is N or W or I or Q or R, we cannot make a similar general statement. This explains the importance of the set C.

(*h*) If a, b, c are real numbers, then the truth set of $ax^2+bx+c=0$ has two real members if $b^2-4ac>0$, has two complex members if $b^2-4ac<0$ and has only real member if $b^2-4ac=0$. This result does not hold if a, b, c are complex, *e.g.*, truth set of $x^2-ix-1=0$ has two complex members through $b^2-4ac>0$.

Experiment 369. Solution of Equations through Mappings and Graphs.

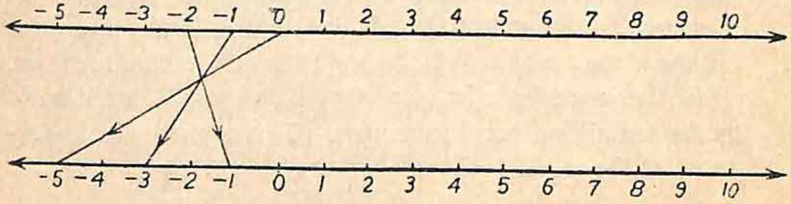
(*i*) Consider the problem of solving $x=2x-6$ or of finding the truth set of $\square=2\square-6$. We consider the mapping which maps every x onto $2x-6$ and we want to find a number which is mapped onto itself.



The image of 1 is -4, the image of 2 is -2, the image of 3 is 0. Continuing this, it is found that the image of 6 is itself. Thus 6 is an invariant element which does not change by mapping. 6 gives the solution of the given equation.

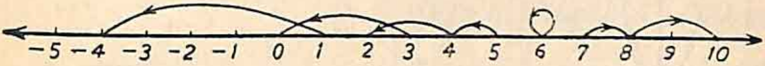
(*ii*) In practice, we need not find the image of all elements. We want a line which is perpendicular to both the number lines. The line corresponding to 3 is inclined, so we try line corresponding to 4, it is also inclined but less, so we try 7, we find it is inclined, but in the opposite direction, so we come back and try 6.

(*iii*) Consider the problem of solving $3x+5=0$ or of $x+2x+5=0$, or of $x=-5-2x$. We consider the mapping which maps x onto $-5-2x$.



We find lines corresponding to -1 and -2 are inclined in opposite directions, so the invariant element lies between -1 and -2. If we try $-\frac{3}{2}$, the line is inclined in a direction opposite to that of -2. Thus the invariant element lies between $-\frac{3}{2}$ and -2. In this way, we can come nearer and nearer to the required solution.

(*iv*) The operation in (*i*) can also be represented geometrically as follows :



(*v*) Consider the equation $x^2-7x+12=0$.

We consider the mapping which maps x to $x^2-6x+12$

	x	0	1	2	3	4	5	6
map of	x	12	7	4	3	4	7	12

We find two invariant elements 3 and 4.

(*vi*) Consider the equation $x^2-x+1=0$.

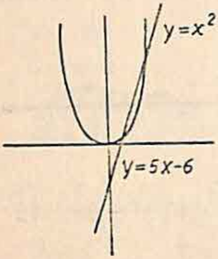
We consider the mapping which maps x to x^2+1.

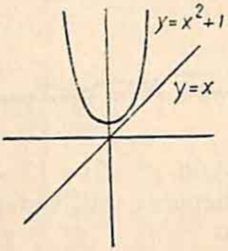
Every negative number is mapped onto a positive number, so a negative number cannot be an invariant element. Every positive number is mapped onto a positive number greater than itself. Thus a positive number also cannot be an invariant element. It appears that this mapping has no invariant element, since the image of every element is greater than itself.

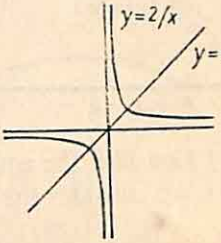
(*vii*) Consider the mapping which maps x onto $\frac{2}{x}$ and try to find the invariant element.

The image of 1 is 2 and the image of 2 is 1 *i.e.*, the image of 1 is greater than itself, while the image of 2 is smaller than itself, so that the invariant point lies between 1 and 2. The image of 1·4 is 1·428... and is greater than itself, while the image of 1·5 is 1·333... and is smaller than itself. Thus the invariant element lies between 1·4 and 1·5. Next we can try 1·41 and 1·42. This process can be continued indefinitely and it can be shown that the process will never terminate. This is a consequence of the irrationality of $\sqrt{2}$ (How ?).

(*viii*) An alternative method of solving equations is through graphs. To solve $3x+5=0$, we draw the graphs of $y=3x+5$ and find where it cuts $y=0$. To solve $x^2-5x+6=0$, we draw the graphs of $y=5x-6$ and $y=x^2$ and find the x-coordinates of the two points of intersection. To solve $x^2-x+1=0$, we draw the graphs of $y=x^2+1$ and $y=x$ and find that they do not intersect.







To solve $x^2=2$, we draw the graphs of $y=x$ and $y=\dfrac{2}{x}$ and find the two points of intersection. By measurement, the children will find $x=1{\cdot}4$ and $-1{\cdot}4$ approximately.

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9

Some Number Theory

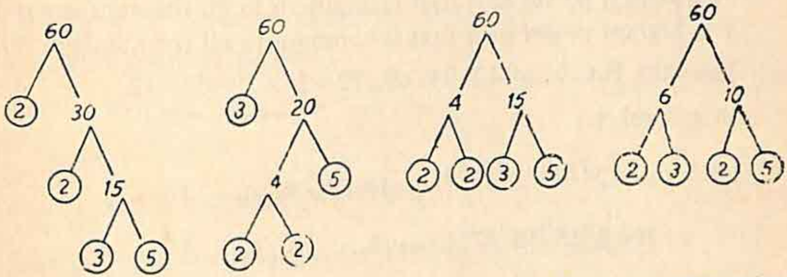
Experiments : 370 to 388

Experiment 370. Factor Trees and Unique Factorization Theorem.

Let the students take numbers and factorize these into factors. Thus they get

$$12=2\times2\times3=2^2\times3=2\times3\times2=3\times2\times2$$
$$18=2\times3\times3=2\times3^2=3\times2\times3=3\times3\times2$$
$$420=2\times2\times3\times5\times7=2^2\times3\times5\times7=3\times2\times5\times2\times7$$
$$648=2\times2\times2\times3\times3\times3\times3=2^3\times3^4$$

Consider the number 60 and different possible ways of factorizing it.



These are called *factor-trees*. Whatever way we factorize, the ultimate prime factors are 2, 3, 2 and 5, though these may arise in different orders. This result holds for every number. This is called *unique factorization theorem* and is of fundamental importance.

Experiment 371. A Way of Representing Numbers.

Every number N can be written uniquely as

$$N=p_1^{n_1}p_2^{n_2}p_3^{n_3}p_4^{n_4}\ldots\ldots$$

where $p_1, p_2, p_3, p_4, \ldots\ldots$ are the prime numbers 2, 3, 5, 7, 11, 13,…… and $n_1, n_2, n_3, n_4, \ldots\ldots$ can be the whole numbers 0, 1, 2, 3, 4, 5,…… We can write even $N=(n_1, n_2, n_3, n_4, \ldots)$

The students can find this representation for a large number of numbers. Thus

$12=(2, 1)$; $18=(1, 2)$; $60=(2, 1, 1)$; $100=(2, 0, 2)$; $169=(0, 0, 0, 0, 0, 2)$; $420=(2, 1, 1, 1)$; $648=(3, 4)$

Let $N=p_1^{n_1}p_2^{n_2}\ldots\ldots\ldots\ldots=(n_1, n_2, \ldots)$

$M=p_1^{m_1}p_2^{m_2}\ldots\ldots\ldots\ldots=(m_1, m_2, \ldots)$

then $N\times M=p_1^{n_1+m_1}p_2^{n_2+m_2}\ldots\ldots=(n_1+m_1, n_2+m_2, \ldots)$

Necessary and sufficient conditions for N to be divisible by M are $n_1 \geqslant m_1$, $n_2 \geqslant m_2$, $n_3 \geqslant m_3, \ldots\ldots$

Experiment 372. Finding H.C.F. of Given Numbers.

Suppose we have to find H.C.F. of 12, 24, 60, 72, we write

$12=2^2\times 3$, $24=2^3\times 3$, $60=2^2\times 3\times 5$, $72=2^3\times 3^2$.

To find the H C.F. we have to find the highest common factor. The highest power of 2 that is common to all the numbers is 2. The highest power of 3 that is common to all the numbers is 1. The highest power of 5 that is common to all the numbers is 0. Thus the H.C.F. of 12, 24, 60, $72=2^2\times 3^1\times 5^0=12$.

In general, let

$$A=2^{a_1}3^{a_2}5^{a_3}7^{a_4}\ldots=(a_1, a_2, a_3, a_4, \ldots)$$

$$B=2^{b_1}3^{b_2}5^{b_3}7^{b_4}\ldots=(b_1, b_2, b_3, b_4, \ldots)$$

$$C=2^{c_1}3^{c_2}5^{c_3}7^{c_4}\ldots=(c_1, c_2, c_3, c_4, \ldots)$$

Then the H.C.F. of A, B, C,… is given by the number obtained by taking the product of the smallest powers of 2, 3, 5,…

Experiment 373. Finding L.C.M. of Given Numbers.

We first factorize numbers A, B, C,… into prime factors as above. To find the least common multiple, we have to find sets of multiples of the numbers. In a multiple of a number, the powers are greater than or equal to the powers in the number. The least common multiple is obtained by taking the product of the largest powers of 2, 3, 5,... Thus

H.C.F. $=$(smallest of $a_1, b_1, c_1, \ldots$, smallest of $a_2, b_2, c_2, \ldots, \ldots$)

L.C.M. $=$(largest of $a_1, b_1, c_1, \ldots$, largest of $a_2, b_2, c_2, \ldots, \ldots$)

Thus consider the numbers

$12=2^2\times3,\ 24=2^3\times3,\ 36=2^2\times3^2,\ 60=2^2\times3\times5,\ 70=2\times5\times7$

Their H.C.F. $=2^1\times3^0\times5^0\times7^0=2$

L.C.M. $=2^3\times3^2\times5^1\times7^1=2520.$

Experiment 374. Relation between H.C.F. and L.C.M.

Let $A=2^{a_1}3^{a_2}5^{a_3}\ldots\ldots$

$B=2^{b_1}3^{b_2}5^{b_3}\ldots\ldots$

$$\text{H.C.F.}=2^{\min.(a_1, b_1)}3^{\min.(a_2, b_2)}5^{\min.(a_3, b_3)}\ldots\ldots$$

$$\text{L.C.M.}=2^{\max.(a_1, b_1)}3^{\max.(a_2, b_2)}5^{\max.(a_3, b_3)}\ldots\ldots$$

Now for two given numbers a_i, b_i, we have

$$\min.(a_i, b_i)+\max.(a_i, b_i)=a_i+b_i$$

Thus for two given numbers A, B

$$A\times B=(\text{H.C.F. of A and B})\times(\text{L.C.M. of A and B})$$

For n numbers we have,

$$\min.(a_1, b_1, c_1, \ldots)\leqslant\frac{a_1+b_1+c_1+\cdots}{n}\leqslant\max.(a_1, b_1, c_1\cdots)$$

so that

H.C.F. $\leqslant$ geometric mean of given numbers $\leqslant$ L.C.M.

Experiment 375. Algorithms for Finding H.C.F. and L.C.M.

We can find factors of given numbers simultaneously as follows :

$$\begin{array}{r|l}
2 & 12,\ 24,\ 36,\ 60,\ 70 \\ \hline
2 & 6,\ 12,\ 18,\ 30,\ 35 \\ \hline
3 & 3,\ 6,\ 9,\ 15,\ 35 \\ \hline
5 & 1.\ 2,\ 3,\ 5,\ 35 \\ \hline
 & 1,\ 2,\ 3,\ 1,\ 7
\end{array}$$

The factors are :

$12=2\times2\times3$

$24=2\times2\times3\times2$

$36=2\times2\times3\times3$

$60=2\times2\times3\times5$

$70=2\times5\times7$

H.C.F. is the product of those divisors which divide any number. In our case the first 2 is the only such divisor, so that H.C.F. $=2$. L.C.M. is the product of those divisors which divide at least one number so that L.C.M.$=2\times2\times3\times5\times2\times3\times7=2520$. This is the usual algorithm for finding L.C.M. What is important to note is that the algorithm for finding L.C.M. also leads to the algorithm for finding H.C.F.

Experiment 376. Irrational Nature of Some Numbers.

The unique factorization theorem gives a simple proof of irrational nature of some numbers.

To prove $\sqrt{2}$ is irrational, if possible, let

$$\sqrt{2}=\frac{A}{B}=\frac{2^{a_1}3^{a_2}5^{a_3}\ldots}{2^{b_1}3^{b_2}5^{b_3}\ldots}$$

or $$2^{2a_1}3^{2a_2}5^{2a_3}\ldots=2(2^{2b_1}3^{2b_2}5^{2b_3}\ldots)$$

but the power of 2 on the L.H.S. is even, while the power of 2 on the R.H.S. is odd ; this is not possible.

The proof can be generalised to show that $R\sqrt{\frac{N}{D}}$ is not a rational number, if N and D are not perfect Rth powers. If possible let

$$R\sqrt{\frac{N}{D}}=\frac{A}{B}$$

then

$$\frac{2^{n_1}3^{n_2}5^{n_3}\ldots}{2^{d_1}3^{d_2}5^{d_3}\ldots}=\left(\frac{2^{a_1}3^{a_2}5^{a_3}\ldots}{2^{b_1}3^{b_2}5^{b_3}\ldots}\right)^R$$

From the unique factorization theorem,

$$n_1+b_1R=d_1+a_1R$$
$$n_2+b_2R=d_2+a_2R$$

....

But these are possible only if n_1, d_1, n_2, d_2,... are all multiples of R *i.e.*, only if N and D are perfect Rth powers. We are assuming that $\frac{N}{D}$ and $\frac{A}{B}$ are in their lowest terms.

Experiment 377. Principle of Divisibility Criteria.

The next five experiments are in continuation of experiment 92. To test whether a given number N is divisible by a number

$$M=p_1^a\, p_2^b\, p_3^c\ldots\ldots$$

where $p_1, p_2, p_3, \ldots$ are primes, we need criteria for divisibility by prime numbers. To test whether N is divisible by p, we usually try to find a number N_1 smaller than N such that N will be divisible by p if and only if N_1 is divisible by p. If necessary, we find a still smaller number N_2 such that N_1 is divisible by p if and only if N_2 is divisible by p. We can continue this process till we get a sufficiently small number whose divisibility can be directly tested.

Experiment 378. Criteria for Divisibility by 7.

Divide 10000000... by 7 and go on dividing till a digit recurs in the quotient. Thus we have :

$$
\begin{array}{rr}
 & 1428571 \\
7\,) & \overline{10000000} \\
 & \underline{7} \\
 & 30 \\
 & \underline{28} \\
 & 20 \\
 & \underline{14} \\
 & 60 \\
 & \underline{56} \\
 & 40 \\
 & \underline{35} \\
 & 50 \\
 & \underline{49} \\
 & 10 \\
 & \underline{7} \\
 & 3
\end{array}
$$

It is obvious that a recurrence must occur since on dividing by 7, the only possible remainders are 0, 1, 2, 3, 4, 5 and 6. If the remainder is 0, the process terminates, otherwise after at most 6 stages, recurrence must occur. In our case the remainder sequence is 1, 3, 2, 6, 4, 5, 1, 3, 2, 6, 4, 5,...

Now let the number to be tested for divisibility be $j\,i\,h\,g\,f\,e\,d\,c\,b\,a$ where the letters denote digits 0 to 9. Now

$$N = a + 10b + 100c + 1000d + 10{,}000e + 100{,}000f + 1{,}000{,}000g + \ldots$$
$$\equiv a + 3b + 2c + 6d + 4e + 5f + g + \ldots \pmod 7$$

The coefficients are the same as the remainders in the division process and these show a cyclic pattern, the cycle being

(1, 3, 2, 6, 4, 5),

Now N is divisible by 7 if and only if

$$N_1 = a + 3b + 2c + 6d + 4e + 5f + g + \cdots$$

is divisible by 7.

Also $N_1 \equiv N_2 \pmod 7$ where

$$N_2 = a + 3b + 2c - d - 3e - 2f + g + \cdots$$

N_2 may be smaller than N_1 and so may be easier to test. Finally we may say that the multiplier cycle for divisibility by 7 is (1, 3, 2, -1, -3, -2). We multiply unit digit of N by 1, tens digit by 3,... and so on to get N_2 and test this smaller number for divisibility by 7.

As an example, let $N=8{,}643{,}837{,}529{,}983{,}580{,}247$

$$\begin{aligned} N_2 &= 7+12+4-24-10+3+24+18-9-6-10+7+9+16 \\ &\quad -3-12-12+8 \\ &= -11+20+13=22 \end{aligned}$$

N_2 is not divisible by 7 and leaves a remainder 1 when divided by 7.

Therefore N is not also divisible by 7 and would leave a remainder 1 when divided by 7.

Experiment 379. Criteria for Divisibility by Other Prime Numbers.

We can develop criteria for divisibility by other prime numbers like 11, 13, 17, 19, 23, 29, 31,... by using the same method.

$$\begin{array}{r} 909 \\ 11\,)\overline{1000000} \\ \underline{99} \\ 100 \\ \underline{99} \\ 1 \end{array}$$

$$\begin{array}{r} 7692307 \\ 13\,)\overline{1000000} \\ \underline{91} \\ 90 \\ \underline{78} \\ 120 \\ \underline{117} \\ 30 \\ \underline{26} \\ 40 \\ \underline{39} \\ 100 \\ \underline{91} \\ 9 \end{array}$$

$$\begin{array}{r} 58823529411764705 \\ 17\,)\overline{1000000} \\ \underline{85} \\ 150 \\ \underline{136} \\ 140 \\ \underline{136} \\ 40 \\ \underline{34} \\ 60 \\ \underline{51} \\ 90 \\ \underline{85} \\ 50 \\ \underline{34} \\ 160 \\ \underline{153} \\ 70 \\ \underline{68} \\ 20 \\ \underline{17} \\ 30 \\ \underline{17} \\ 130 \\ \underline{119} \\ 110 \\ \underline{102} \\ 80 \\ \underline{68} \\ 120 \\ \underline{119} \\ 1 \end{array}$$

The remainders for 11 are 1, 10, 1, 10, ...

The multiplier cycle is (1, 10) or (1, −1)

The remainders for 13 are (1, 10, 9, 12, 3, 4, 1, 10, ...)

The multiplier cycle is (1, 10, 9, 12, 3, 4)

or (1, −3, −4, −1, 3, 4)

The remainders for 17 are : 1, 10, 15, 14, 4, 6, 9, 5, 16, 7,
2, 3, 13, 11, 8, 12, 1, 10, ...

The multiplier cycle is (1, 10, 15, 14, 4, 6, 9, 5, 16, 7, 2,
3, 13, 11, 8, 12)

or (1, −7, −2, −3, 4, 6, −8, 5, −1, 7, 2, 3, −4, −6, 8, −5)

We may note a certain symmetry in the multiplier cycles *e.g.*,
cycle for 7 is (1, 3, 2, −1, −3, −2)
cycle for 11 is (1, −1)
cycle for 13 is (1, −3, −4, −1, 3, 4)
cycle for 17 is (1, −7, −2, −3, 4, 6, −8, 5, −1, 7, 2, 3, −4, −6, 8, −5)

In each case the cycle consists of an even number of terms and the terms of the second half are negatives of the term of the first half.

The students may investigate whether this pattern always holds. If at any stage in the division process, we get a remainder one less than the divisor, this pattern will hold after that and we need not continue the division process further.

Experiment 380. Criterion for Divisibility by Composite Numbers.

Normally one develops criteria for divisibility by prime numbers only and if $N=p_1, p_2, p_3\ldots\ldots$, we can test separately for divisibility by p_1, p_2, p_3, etc.

However this method will not determine remainder when the given number is not divisible by N.

Moreover if $N=p_1^{n_1}p_2^{n_2}\ldots\ldots$, then if the given number is divisible by p_1^2, we may have to divide the given number by p_1 and then test whether the quotient is divisible by p_1. This will be time-consuming process. Consider criterion for divisibility by 12. The remainders here are

1, 10, 4, 4, 4,......

$$\begin{array}{r} 833 \\ 12\,\overline{)\,10000} \\ \underline{96} \\ 40 \\ \underline{36} \\ 40 \\ \underline{36} \\ 40 \end{array}$$

Thus there is here both a non-recurring part and a recurring part and we can write the multiplier cycle as 1, −2, (4)

To test whether a given number $N=a_6\,a_5\,a_4\,a_3\,a_2\,a_1$ is divisible by 12, we find whether

$$N_1=a_1-2a_2+4(a_3+a_4+a_5+a_6)$$

is divisible by 12. If N is not divisible by 12, N and N_1 will give same remainders on division by 12. It will be found that

(*i*) when N consists of powers of 2 along with powers of primes other than 2, we get a non-recurring part

(*ii*) when N consists of powers of 5 along with powers of primes other than 5, we again get a non-recurring part

(*iii*) when N consists of powers of primes other than 2 and 5, we get a recurring part only

(*iv*) when N consists of powers of 2 and 5 alone, we can test the divisibility easily otherwise [see experiment 80].

Experiment 381. An Alternative Approach to Obtaining Divisibility Criteria.

Let

$$N = a_0\, a_1\, a_2 \ldots\ldots a_n$$

then we know that

$$\begin{aligned} N &= a_0 10^n + a_1 10^{n-1} + a_2 10^{n-2} + \ldots\ldots + a_n \\ &= 10(a_0 10^{n-1} + a_1 10^{n-2} + \ldots + a_{n-1} - k a_n) + (10k+1)a_n \\ &= 10N_1 + (10k+1)a_n \end{aligned}$$

Now choose k so that $10k+1$ is divisible by the prime number p, then N will be divisible by p if N_1 is divisible by p and N_1 is smaller than N.

Now the choice of k will be possible except when p is 2 or 5. Thus this method of reduction of N to N_1 will not be available from $p=2$ or 5 but this is not a serious draw-back since simple criteria for divisibility by 2 or 5 are available.

We want $10k+1=ps$.

If p ends in 1, we choose $s=1$

If p ends in 3, we choose $s=7$

If p ends in 7, we choose $s=3$

If p ends in 9, we choose $s=9$.

Thus we have the table

p	7	11	13	17	19	23	29	31	37	41	43	47
s	3	1	7	3	9	7	9	1	3	1	7	3
k	2	1	9	5	17	16	26	3	11	4	30	14

As an example, let N=390224, $p=29$, $k=26$

$$\begin{aligned} N &= 390224 \\ N_1 &= 39022 - 26 \times 4 = 38918 \\ N_2 &= 3891 - 26 \times 8 = 3683 \\ N_3 &= 368 - 26 \times 3 = 290 \end{aligned}$$

In this case the reduction is by one digit at a time and so the process may be a slower process.

We may also note that in this case, N and N_1 need not give the same remainder on division by p, as was the case in the multiplier cycle pattern.

Experiment 382. Patterns in Odds and Evens.

Arrange all natural numbers in two classes, the class of odd numbers and the class of even numbers as follows :

O	E
1	2
3	4
5	6
7	8
9	10
11	12
13	14
...	...
...	...

We note that
any odd number+any odd number=an even number
any odd number+any even number=an odd number
any even number+any odd number=an odd number
any even number+any even number=an even number

We write this as following addition table :

+	*O*	*E*
O	*E*	*O*
E	*O*	*E*

Similarly, we get the multiplication table

×	*O*	*E*
O	*O*	*O*
E	*O*	*E*

We also get the following table for squares, cubes etc.

	Square	Cube	4th power
O	O	O	O
E	E	E	E

We deduce that all powers of odd numbers are odd and all powers of even numbers are even.

If n is an odd number, n^2, n^3, n^4,..., are all odd numbers.

If n is an even number, n^2, n^3, n^4,..., are all even numbers.

What about n^2+n?

If n is odd, n^2 is odd and the sum of two odd numbers is even. If n is even, n^2 is even and the sum of two even numbers is even. Thus n^2+n is always even.

What about n^3+n^2+n?

If n is odd, it is sum of three numbers and so is odd. If n is even, it is sum of three even numbers and so it must be even.

In the same way it can be shown that n^2+n+1 is always odd, $n^4+n^3+n^2+n$ is always even, $n^4+n^3+n^2+n+1$ is always odd. The children can now generalise that

$n^{2m}+n^{2m-1}+\ldots\ldots+n$ is always even

$n^{2m}+n^{2m-1}+\ldots\ldots+n+1$ is always odd.

$n^{2m+1}+n^{2m}+\ldots\ldots+n$ is odd when n is odd and is even when n is even. $n^{2m+1}+n^{2m}+\ldots\ldots+n+1$ is even when n is odd and is odd when n is even. What about n^2+n+41? Since n^2+n is always even, n^2+n+41 is always odd. All prime numbers except 2 are also always odd, but all odd numbers are not prime *i.e.*, for any number greater than 2 to be a prime number, it is necessary that it should be an odd number ; however this is not sufficient.

Let the children check that n^2+n+41 is a prime number for $n=1, 2, 3, 4, \ldots, 39$ but for $n=40$ it is a composite number.

What about n^2-n ? If n is even, it is the difference of two even numbers and so is even. If n is odd, it is the difference of two odd numbers and so is even. Thus n^2-n is always even when $n>1$. It can also be checked that n^2-n+41 is prime for $n=1, 2, \ldots, 40$ and is composite when $n=41$.

We have got expressions above which are always even or always odd. Children can get more such expressions *e.g.*, $n^{2m}+n^{2m-1}+\ldots+n$ $+$any even number is always even, $n^{2m}+n^{2m-1}+\ldots+n+$any odd number is always odd.

What about $3n^2+n$? If n is odd, it is sum of two odd numbers and so is even. If n is even, it is sum of two even numbers and so is again even. Thus $3n^2+n$ is always even and the result would remain true if 3 is replaced by any odd number. The children can now generalise that $an^{2m}+bn^{2m-1}+\ldots+kn+$any even number is always even if $a, b, c, \ldots, k$ are all odd and $an^{2m}+bn^{2m-1}+\ldots+kn+$an odd number is always odd if $a, b, c, \ldots, k$ are all odd.

Experiment 383 Patterns in Multiples and Non-multiples of Three.

We have discussed in the last experiment patterns in multiples of 2 (*viz.*, even numbers) and in non-multiples of 2 (*i.e.* odd numbers) and found expressions which were always even or always odd. We now want to find similar expressions in the case of multiples of 3.

Any given number is divisible by 3 or leaves remainder 1 or 2 on division by 3. Thus here these are three classes of numbers, *viz.*, numbers of the form $3n$ or $3n+1$ or $3n+2$. We denote these classes by A, B, C.

A	B	C
1	2	3
4	5	6
7	8	9
10	11	12
13	14	15
...	...	...

Now we define the sum of two classes as that class to which the sum of any member of the first class and any member of the second class belongs. We similarly define the product of two classes. Then we easily get the following tables :

+	*A*	*B*	*C*
A	*B*	*C*	*A*
B	*C*	*A*	*B*
C	*A*	*B*	*C*

×	*A*	*B*	*C*
A	*A*	*B*	*C*
B	*B*	*A*	*C*
C	*C*	*C*	*C*

Consider $n(n+1)(n+2)$.

If n belongs to class A, $n+1$ belongs to class B, $n+2$ belongs to C and the product belongs to class C.

If n belongs to class B, $n+1$ belongs to class C, $n+2$ belongs to class A and the product belongs to class C.

If n belongs to class C, $(n+1)$ belongs to class A, $(n+2)$ belongs to class B and the product belongs to class C.

Thus whatever be n, $n(n+1)(n+2)$ belongs to class C.
Thus for all values of n, $n(n+1)(n+2)$ is a multiple of 3.
Thus n^3+3n^2+2n is always a multiple of 3.

Since $3n^2$ is a multiple of 3, we have n^3+2n is always a multiple of 3. We can add or subtract a multiple of 3 and the resulting expression will always be a multiple of 3. Thus n^3+5n, n^3+8n, n^3-n, $4n^3+5n$, $4n^3+2n$, $2n^3+n$, n^3+3n^2+5n, n^3+3n^2+8n, n^3-3n^2+5n etc. are all multiples of 3, whatever be the value of n.

If we add 1 to any member of class C, we get a member of class A.

Thus n^3+3n^2+2n+1, n^3+2n+1, n^3+5n+1, n^3-n+1, $2n^3+n+1$ etc. are all such that when divided by 3, they leave 1 as remainder.

Similarly if we add 2 to any member of class C or subtract 1 from it, we get a member of class B so that the expressions

$$n^3+3n^2+2n-1,\ n^3+2n+2,\ n^3-n+2,\ n^3-n+5 \text{ etc.}$$

are all such that when divided by 3, they leave 2 as remainder. The children will have great fun constructing hundreds of examples of expressions belonging to either class A or B or C. They may also note that

(*i*) If an expression belongs to a certain class and if n is replaced by $n+1$ or $n+2$ or $2n+1$ or n^2 or n^3 or n^2+1 etc., it still continues to belong to that class.

(*ii*) The sum and the product of two expressions belonging to class C both belong to class C.

(*iii*) The sum and product of two expressions belonging to class B both belong to class A.

(*iv*) The sum of two expressions belonging to class A belongs to class B and the product of two expressions belonging to class A belongs to class A.

The children can deduce many other rules like this with the help of the above two addition and multiplication tables and from expressions belonging to class A or B or C. They can verify the results for particular values of n. They should understand also that verification of the fact that n^3+2n is always a multiple of 3 would take an infinite amount of time, while above they have a simple proof that it would always be so and they have got not one such expression, but they can make any number of such expressions.

The children can now form 5 classes of numbers as follows :

A	B	C	D	E
1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
...	...	...	...	...
...	...	...	...	...

They can form addition and multiplication tables for these. They can show $n(n+1)\,(n+2)\,(n+3)\,(n+4)$ always belongs to the class E and they can construct large number of expressions belonging to class A or B or C or D or E.

This can repeat the process with 7 classes.

Experiment 384. Patterns in Fibonacci Sequence.

Consider the sequence published in 1202 by Leonardo of Pisa (also known as Fibonacci)

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377,...

in which each term is equal to the sum of the preceding two terms.

Let the students multiply two consecutive terms, two terms with one gap. two terms with two gaps, two terms with three gaps, two terms with four gaps etc. and discover the following patterns, where u_n denotes the *n*th term.

(*i*) $u_n = u_{n-1} + u_{n-2}$
(*ii*) $u_n \times u_{n+1} = u_{n-1} \times u_{n+2} \pm 1$
(*iii*) $u_n \times u_{n+1} = u_{n-2} \times u_{n-3} \pm 2$
(*iv*) $u_n \times u_{n+1} = u_{n-3} \times u_{n+4} \pm 6$
(*v*) $u_n \times u_{n+1} = u_{n-4} \times u_{n+5} \pm 15$
(*vi*) $u_{n-1} \times u_{n+1} = u_n^2 \pm 1^2$
(*vii*) $u_{n-2} \times u_{n+2} = u_n^2 \pm 1^2$
(*viii*) $u_{n-3} \times u_{n+3} = u_n^2 \pm 2^2$
(*ix*) $u_{n-4} \times u_{n+4} = u_n^2 \pm 3^2$
(*x*) $u_{n-5} \times u_{n+5} = u_n^2 \pm 5^2$
(*xi*) $u_n^2 + u_{n+1}^2$ is a Fibonacci number
(*xii*) $u_1 + u_2 + \ldots + u_n = u_{n+2} - 1$.

Let the students discover more such patterns. Let them write more such Fibonacci sequences where each term is the sum of the preceding two terms (the first two terms are arbitrary) or where each term is the sum of the preceding three terms etc. and find similar patterns in these sequences.

Experiment 385. The Usual Algorithm for Finding H.C.F.

When we divide a natural number N by another smaller natural number D, we get a quotient Q and a remainder R such that

$$N = DQ + R,\ O \leqslant R < Q,\ D = 1, 2, \ldots$$

Let children take specific values, say $N=50$, $D=12$; $N=504$, $D=13$; $N=1092$, $D=12$; $N=525$, $D=5$ and find the values of Q and R in each case.

This equation gives the justification for the usual algorithm of finding H.C.F. of a set of natural numbers. Consider the specific example of finding H.C.F. of 42 and 77. The usual algorithm gives calculations given on the right. These calculations can be done in 3 steps as follows :

$$\begin{array}{l}
42\overline{)\,77\,}\,(1 \\
\quad\ \ \underline{42} \\
\quad\ \ 35\overline{)\,42\,}\,(1 \\
\qquad\quad \underline{35} \\
\qquad\quad 7\overline{)\,35\,}\,(5 \\
\qquad\qquad \underline{35} \\
\qquad\qquad\ \times
\end{array}$$

$$\begin{array}{lll}
42\overline{)\,77\,}\,(1 & \quad 35\overline{)\,42\,}\,(1 & \quad 7\overline{)\,35\,}\,(5 \\
\quad\ \ \underline{42} & \quad\quad\ \ \underline{35} & \quad\quad \underline{35} \\
\quad\ \ 35 & \quad\quad\ \ \ 7 & \quad\quad\ 0
\end{array}$$

These give rise to three equations

$77=1\times42+35$ (1)

$42=1\times35+7$ (2)

$35=5\times7+0$ (3)

From (3), 7 is divisor of 35

From (2), 7 is a divisor of 7 and 35 both, so it is a divisor of 42

From (1), 7 is divisor of 35 and 42 both, so that it is a divisor of 77.

Therefore 7 is a common divisor of 42 and 77.

Is it the highest common divisor ? Suppose another higher number could be the divisor of 42 and 77, then from (1), it must be a divisor of 35 also and from (2) it must be a divisor of 7 also.

Let the children do a number of particular problems and justify each in the above way and arrive at a more general formulation. Let N and M be the two natural numbers and let $N>M$, then the algorithm gives

$$\begin{aligned}
N &= M\,Q_1+R_1 \\
M &= R_1Q_2+R_2 \\
R_1 &= R_2Q_2+R_3 \\
R_2 &= R_3Q_3+R_4 \\
&\cdots\cdots\cdots\cdots\cdots \\
R_{k-1} &= R_kQ_k+R_{k+1} \\
R_k &= R_{k+1}Q_{k+1}+0
\end{aligned}$$

We start first with the last equation. From this equation we find R_{k+1} divides R_k and from the last but one, it then divides R_{k-1} and we continue backwards till we find from the first two equations that it divides M and N. To show it is the highest common divisor, we start with the first equation and suppose that a larger number can be a common divisor of both N and M. If this larger number divides both N and M, it also divides R_1 and then from the second equation it divides R_2 and from the third equation it divides R_3 and so on till from the last but one equation, we find that its divides R_{k+1}. Thus any common divisor must divide R_{k+1}.

Thus the division algorithm leads to the H.C.F. or the H.C.D. of the two given numbers.

Experiment 386. H.C.F. and L.C.M. in the Domain of Integers.

We define H.C.F. of a set of two or more numbers as that number which divides every number of the set and which is such that every number that divides all the numbers of the set divides this number *e.g.* H.C.F. of 20, 30, 40 is 10, since 10 divides all these numbers and every number (like 2 or 5) which divides all these numbers also divides 10.

We define L.C.M. of a set of two or more numbers as that number which is such that every number of the set divides this number of the set *e.g.* L.C.M. of 20, 30, 40 is 120 since all the three numbers divide 120 and 120 divides all those numbers which are divisible by 20, 30, 40 (like 240, 360, 480,...).

Let us apply these definitions to integers. What is the H.C.F. of −20, −30, −10, 0, 10, 20 ? Both 10 and −10 satisfy the conditions, since the quotients can be integers. Each of these divides all the numbers of the set and those numbers which divide all numbers of the set (like −2, −5, 2, 5) divide 10 and −10.

Thus in the case of integers, H.C.F. of a set of integers is not unique, but there can be two H.C.F's which differ in sign only. We say that the H.C.F. is unique except for sign. To find the H.C.F. of a set of integers, we find the H.C.F. of the absolute values of non-zero members of the set and then this value together with its additive inverse gives the two H.C.F.'s.

Next consider the L.C.M. of a set of non-zero integers *e.g.* −20, −30, −10, 10, 20. The integers 60 and −60 are divisible by all of them and all numbers which are divisible by all of them (like 120, -120, 240, -240,...) are divisible by 60 and -60.

Thus the L.C.M. of a set of non-zero integers is unique except for sign.

To find the L.C.M. of a set of non--zero integers, we find the L.C.M. of their absolute values. This number together with its negative gives the two L.C.M.'s of the given set of numbers.

Experiment 387. H.C.F. and L.C.M. in Finite Arithmetics.

What is the H.C.F. of 2, 3, 4 in the system $[\{0, 1, 2, 3, 4\}, +_5, \times_5]$?

1 is a divisor of 0, 1, 2, 3, 4, since

$$\frac{0}{1}=0,\ \frac{1}{1}=1,\ \frac{2}{1}=2,\ \frac{3}{1}=3,\ \frac{4}{1}=4$$

2 is a divisor of 0, 1, 2, 3, 4 since

$$\frac{0}{2}=0,\ \frac{1}{2}=3,\ \frac{2}{2}=1,\ \frac{3}{2}=4,\ \frac{4}{2}=2$$

3 is a divisor of 0, 1, 2, 3, 4 since

$$\frac{0}{3}=0,\ \frac{1}{3}=2,\ \frac{2}{3}=4,\ \frac{3}{3}=1,\ \frac{4}{3}=3$$

4 is a divisor of 0, 1, 2, 3, 4 since

$$\frac{0}{4}=0,\ \frac{1}{4}=4,\ \frac{2}{4}=3,\ \frac{3}{4}=2,\ \frac{4}{4}=1$$

Thus 1, 2, 3, 4 are all divisors of 2, 3, 4 and each of these can be regarded as the highest common divisor or H.C.F., since every other divisor divides each. Thus for any subset of {0, 1, 2, 3, 4}, each of the numbers 1, 2, 3, 4 can be regarded as the H.C.F. Similarly for any subset not including 0, each of the numbers can be regarded as the L.C.M.

The children can easily show that the similar results will apply for the system $(\{0, 1, 2, 3, 4,..., p-1\}, +_p, \times_p)$ where p is a prime number.

Consider now the system $[\{0, 1, 2, 3, 4, 5\}, +_6, \times_6]$

Divisors of 0 are 1, 2, 3, 4, 5 (verify)
Divisors of 1 are 1, 5 (verify)
Divisors of 2 are 1, 5 (verify)
Divisors of 3 are 1, 5 (verify)
Divisors of 4 are 1, 5 (verify)
Divisors of 5 are 1, 5 (veryfy)

From any subset, the common divisors are 1, 5 and each divides the other. Thus both can be regarded as giving H.C.F. of any subset.

Multiple of 0 is 0
Multiples of 1 are 0, 1, 2, 3, 4, 5
Multiples of 2 are 0, 2, 4
Multiples of 3 are 0, 3
Multiples of 4 are 0, 2, 4
Multiples of 5 are 0, 1, 2, 3, 4, 5
L.C.M. of 2 and 4 are both 2 and 4
L.C.M. of 3 and 5 is 3
L.C.M. of 3, 4, 5 is 0.

Children can find the H.C.F.'s and L.C.M.'s in other finite systems in which the number of elements is not prime.

Experiment 388. H.C.F. and L.C.M. in Rational Number System.

Consider first the set of positive rational numbers. Every positive rational number is a divisor of every other positive rational number in the sense that the quotient is always a positive rational number. Thus given a subset of positive rational numbers, every positive rational number is a divisor and there is no greatest divisor. Similarly every positive rational number is a multiple of any other positive rational number in the sense that the quotient of these would be a positive rational number and there is not least common multiple. Thus in this sense, the H.C.F. and L.C.M. are not significant concepts.

However we modify the definition of divisibility and say that a positive rational number is divisible by another only if the quotient is an integer.

Consider a subset of positive rational numbers say

$$\frac{10}{12}, \frac{20}{36}, \frac{30}{72}, \frac{40}{48}$$

The H.C.F. of numerators is 10 and the L.C.M. of denominators is 144. Consider the number $\frac{10}{144}$. It divides each of the four numbers and every other divisor of these numbers like $\frac{5}{288}$, $\frac{5}{144}$, $\frac{5}{432}$, $\frac{1}{576}$ divides this number. Thus $\frac{10}{144}$ can be considered the H.C.F. of these fractions.

Again the L.C.M. of the numerators is 120 and the H.C.F. of denominators is 12. Now consider $\frac{120}{12}=10$. It is an integral multiple of each of these fractions and every other integral multiple of each of these fractions like 20, 30, 40,...would be an integral multiple of 10.

Thus to find the H.C.F. of a set of positive fractions, we find the H.C.F. of numerators and divide it by the L.C.M. of denominators and to find the L.C.M. of a set of positive fractions, we find the L.C.M. of numerators and divide it by the H.C.F. of denominators.

The children can verify this rule in a number of examples and then try to establish it.

For the set of both positive and negative fractions when quotients can be both positive and negative integers, we find the H.C.F. or L.C.M. of absolute values and then consider both positive and negative signs of the resulting numbers.

The children may also investigate the H.C.F. and L.C.M. of positive real numbers when quotients are restricted to be natural numbers or rational numbers only.

10

Recognizing Multinomial Patterns

Experiments : 389 to 394

Experiment 389. What is a Multinomial ?

Consider $1^2+2.1+2$

$2^2+2.2+3$

$3^2+2.3+4$

$4^2+2.4+5$

$\cdots\cdots\cdots\cdots\cdots\cdots$

Can the children find a pattern here ? Can they find an expression of which all these are special cases. Each expression contains three terms. The first terms are 1^2, 2^2, 3^2, 4^2, ...and they are all obtained from n^2 by putting $n=1, 2, 3, 4,\ldots$ The second terms are 2.1, 2.2, 2.3, 2.4, ... These are obtained from $2.n$ by giving particular values $n=1, 2, 3, 4, \ldots$ Similarly the third terms are obtained from $n+1$ by putting $n=1, 2, 3, 4,\ldots$

Therefore the general expression of which these are particular cases is

$$n^2+2n+(n+1) \quad , \quad (n=1, 2, 3, 4,\ldots)$$

This is obviously a more compact form for giving all these expressions. We could have also written it as

$$x^2+2x+(x+1) \quad , \quad (x=1, 2, 3, 4, \ldots)$$

or as $$y^2+2y+(y+1) \quad , \quad (y=1, 2, 3, 4, \ldots)$$

Each expression is the sum of three terms. An expression consisting of three terms is called a *trinomial*. Similarly an expression containing two terms is called a *binomial*, an expression containing four terms is called a *quadranomial* and in general an expression containing many terms is called a *multinomial.*

Experiment 390 Examples of Pattern Formation.

(*i*) We give more examples of the same type :

SPECIAL CASES			GENERAL CASE
$2.1+3$,	$2.2+3$,	$2.3+3$,	$\ldots 2.n+3$
1^3+2^2+3 ,	2^3+3^2+4,	3^3+4^2+5,	$\ldots n^3+(n+1)^2+(n+2)$
$1.2+2.3+3.4$,	$2.3+3.4+4.5$,	$3.4+4.5+5.6$,	$\ldots n(n+1)+(n+1)(n+2)+(n+2)(n+3)$
$2.1^3+3.1^2+4.1+6$,	$2.2^3+3.2^2+4.2+6$,	$2.3^3+3.3^2+4.3+6$,	$\ldots 2n^3+3n^2+4n+6$

Many more such examples can be constructed.

(*ii*) Consider

$$\begin{array}{r}
2.3 + 3.5 \\
3.4 + 4.8 \\
4.6 + 6.10 \\
8.15+15.20 \\
11.30+30.20
\end{array}$$

Here the only pattern we see is that the second factor of the first term is the same as the first factor of the second term. Thus a general pattern is

$$a.b+b.c$$

and all the above expressions are special cases of this general expression.

(*iii*) We give some more examples.

$$\left.\begin{array}{l}
2.3 + 3.5 + 5.11 \\
5.8 + 8.11+11.20 \\
7.20+20.31+31.40
\end{array}\right\} \text{are special cases of } a.b+b.c+c.a$$

$$\left.\begin{array}{l}
2^2+ 3^2 +5^2 \\
3^2+ 8^2 +9^2 \\
5^2+10^2 +9^2
\end{array}\right\} \text{are special cases of } a^2+b^2+c^2.$$

$$\left.\begin{array}{l}
3^3+ 4^2 +5 \\
2^3+ 3^2 +4 \\
(-10)^3+(-9)^2+(-8)
\end{array}\right\} \text{are special cases of } a^3+(a+1)^2+(a+2)$$

$$\left.\begin{array}{l} \left(\frac{3}{5}\right)^3+\left(\frac{5}{7}\right)^2+\left(\frac{7}{11}\right) \\ \left(\frac{1}{3}\right)^3+\left(\frac{3}{5}\right)^2+\left(\frac{5}{9}\right) \\ \left(\frac{-5}{-3}\right)^3+\left(\frac{-3}{-1}\right)^2+\left(\frac{-1}{3}\right) \end{array}\right\} \begin{array}{l} \text{are special cases of } \left(\frac{a}{a+2}\right)^3 \\ \qquad +\left(\frac{a+2}{a+4}\right)^2+\left(\frac{a+4}{a+8}\right) \end{array}$$

$$\left.\begin{array}{l} (2, 3)+(3, 5)+(5, 7) \\ (4, 5)+(5, 7)+(7, 9) \\ (5, 6)+(6, 8)+(8, 10) \end{array}\right\} \begin{array}{l} \text{are special cases of} \\ (a, a+1)+(a+1, a+3)+(a+3, a+5) \end{array}$$

$$\left.\begin{array}{l} (-2, -4, -6)(0, 3, 6)(2, 6, 10) \\ (-5, -7, -9)(-3, 0, 3)(-1, 3, 7) \\ (-8, -10, -12)(-6, -3, 0)(-4, 0, 4) \end{array}\right\} \begin{array}{l} \text{are special cases of} \\ (a, a-2, a-4)(a+2, \\ a+5, a+8)(a+4, \\ a+8, a+12) \end{array}$$

Experiment 391. Use of Pattern Formation.

The examples show that we can use negative integers, fractions, ordered pairs, ordered triplets etc., and we should use the minimum number of symbols in pattern formation.

This pattern recognition and pattern formation will give the children a correct perspective about the use of symbols in algebra.

This will also give them a feeling for abstraction in mathematics.

Experiment 392. Distinction Between Multinomial and Polynomial.

An expression which is the sum of a number of terms is called a multinomial, while a polynomial is an expression of the form

$$a_0x^r+a_1x^{r-1}+\ldots+a_r$$

involving various powers of the 'indeterminate' or 'unknown' or 'variable' x. Here x can take different values from a substitution set and $a_0, a_1, a_2, \ldots, a_r$ are certain constants. Every polynomial is a multinomial, but every multinomial is not a polynomial.

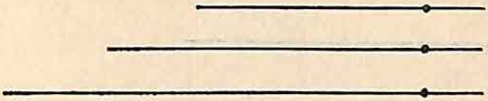
We however consider

$$a^2+ab+b^2+bc+ca$$

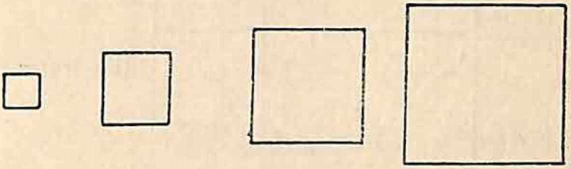
as a polynomial in three variables a, b, c.

Experiment 393. Geometrical Illustrations for Multinomials.

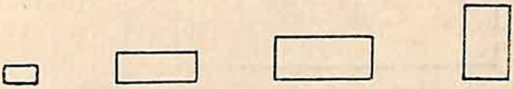
$a+2$



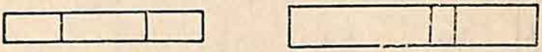
a^2



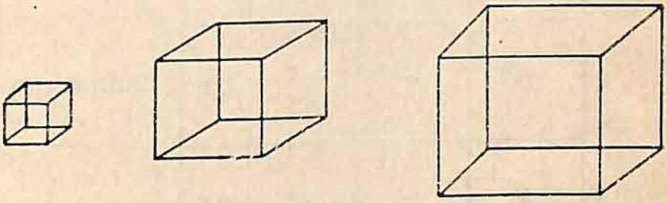
ab



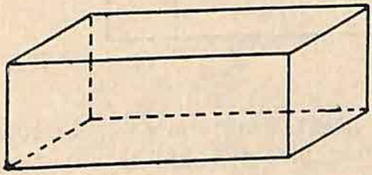
$ab+ac+ad$



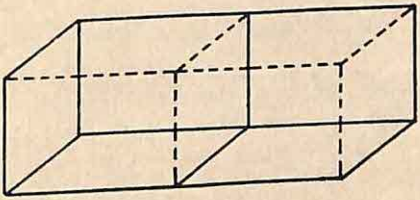
a^3



abc

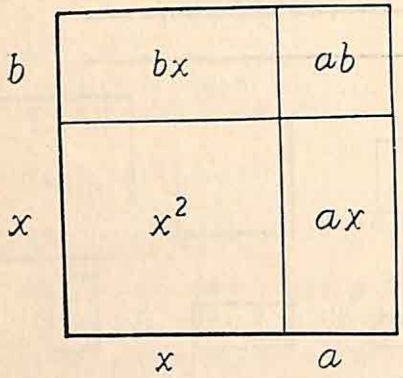


$abc+abd$

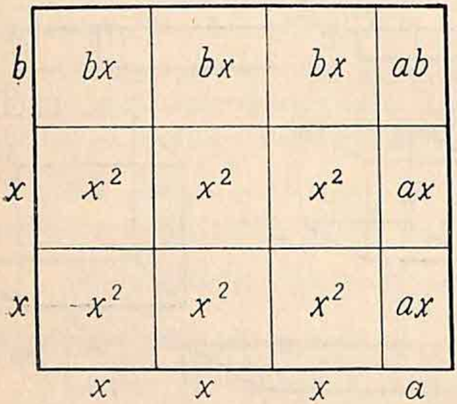


Let the children be given different multinomials involving one, two or three unknowns and let them illustrate these.

Experiment 394. Geometrical Illustrations of Multiplication of Binomials and Trinomials.



Illustrates $(x+a)(x+b)$
$=x^2+ax+bx+ab$



illustrates $(3x+a)(2x+b)$
$=6x^2+2ax+3bx+ab$

Children can also make three-dimensional models to illustrate the multiplication of three binomials and two or three trinomials.

11

Graphs

Experiments : 395 to 408

Experiment 395. Uses of Graphs.

Drawing of graphs should have an important place in school mathematics, specially at the elementary and middle school level, for the following reasons :

(*i*) Children love to draw graphs.

(*ii*) Through graphs, the teaching of mathematics can be intimately related with the teaching of other subjects like history, geography, social studies, physics, chemistry, biology etc.

(*iii*) Through the use of graphs, children can be made conscious of their surroundings.

(*iv*) The graphs make a visual impact on the minds of the children and thus make it easier for them to understand and remember both physical and mathematical laws.

(*v*) Individual children can be given projects of which the final results can be expressed in graphs and such projects can encourage a great deal of initiative and creativity in children.

(*vi*) Children can recognize mathematical and physical patterns through graphs.

(*vii*) Graphs can help in the study of area, volume, mensuration and even in introducing new concepts like that of probability.

(*viii*) The teaching of arithmetic which mostly consists of study of two special types of relations represented by constant slope and constant product graphs, is considerably simplified with the help of graphs.

(*ix*) The concepts of slope, continuity, function, maximum—minimum can be introduced at an elementary level through graphs.

(*x*) Negative numbers can be introduced through graphs.

(*xi*) The concepts of errors of measurements can be introduced through graphs.

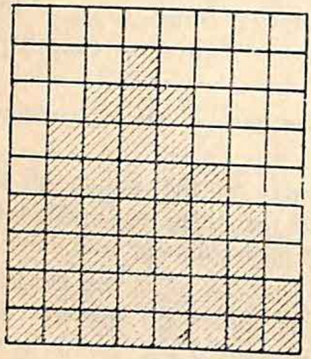
(*xii*) The concept of scientific laws can be introduced through graphs.

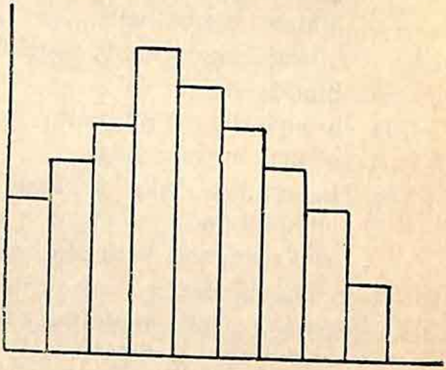
(*xiii*) Since most of the facts of modern society are represented through graphs, every citizen has to know how to draw graphs and how to interpret them.

(*xiv*) The concepts of coordinate geometry and statistics can also be introduced through graphs.

Experiment 396. Block Graphs.

These can be drawn by shading on graph paper or by pasting coloured paper strips on ordinary or graph paper.





Some typical block graphs which can be drawn by the children are the following :

(*i*) Number of families against number of children in each family, for all the children of the class.

(*ii*) Number of children against sizes of shoes they wear.

(*iii*) Number of children in different classes of the school.

(*iv*) Number of children present on different days of the week.

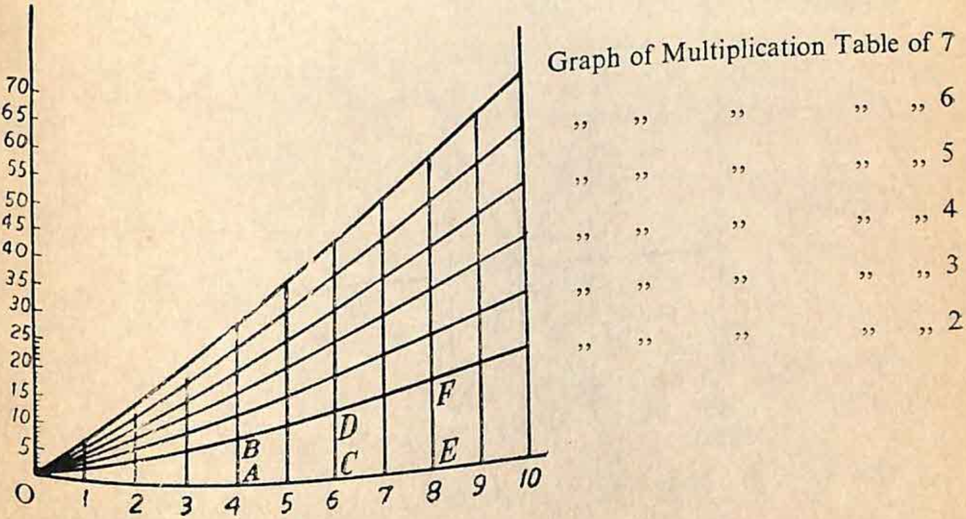
(*v*) Number of years of rule of different kings.

(*vi*) Heights of different mountains in the world.

(*vii*) Lengths of different rivers of the world.
(*viii*) Distance of important stations from your station.
(*ix*) Number of books of different subjects in the library.
(*x*) Number of books bought in different years in the library.
(*xi*) Average lives of different animals.
(*xii*) Number of children of the class born in different months of the year.
(*xiii*) Number of members of different political parties in the Assembly.
(*xiv*) Number of vehicles of different types (rickshaw, cycle, tonga, car, truck, bus etc.) passing in front of the school or on a busy road in an hour.
(*xv*) Lengths of air routes from New Delhi to important cities of the world.
(*xvi*) Diameters of different planets.
(*xvii*) Number of children having different heights.
(*xviii*) Number of children having different weights.

There can be different projects for different children or groups of children in the class. Each project involves collection of data from other children or from some books or from maps, finding a proper scale and then drawing of the graph. Each project may imply a total involvement of a group of children in collective work which is a good training in itself.

Experiment 397. **Graphs of Multiplication Tables.**



For multiplication table of 2, we draw the points :

(1, 2), (2, 4), (3, 6), (4, 8), (5, 10), (6. 12), (7, 14), (8, 16), (9, 18), (10, 20) and join them by a straight line which is graph of multiplication table of 2.

For multiplication table of 3, we draw the points :

(1, 3), (2, 6), (3, 9), (4, 12), (5, 15), (6, 18), (7, 21), (8, 24), (9, 27), (10, 30) and join them by another straight line which is graph of multiplication table of 3.

We find that the graph of multiplication table of 3 is steeper than the graph of multiplication table of 2. This is expressed by stating that the graph of multiplication table of 3 has greater slope. We find that

$$\frac{AB}{OA}=\frac{CD}{OC}=\frac{EF}{OE}=\dots\dots\dots\dots=2$$

This constant ratio is called the *slope* of the line. We also say that triangles OAB, OCD, OEF etc. are *'similar' triangles*. The children can draw different straight lines through O and find their slopes. They can also find slopes of roads (by using spirit level) or of sliding slopes they use for games.

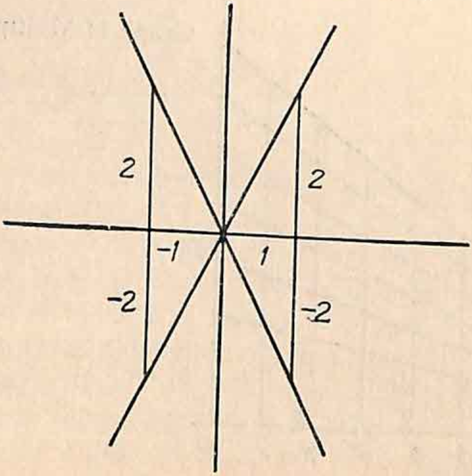
The children can draw graphs of multiplication table upto 10 and note that these are all constant slope graphs.

They can extend the graphs on the left to get

$$2\times(-1)=-2,\ 2\times(-2)=-4,\ 2\times(-3)=-6$$
$$3\times(-1)=-3,\ 3\times(-2)=-6,\ 3\times(-3)=-9$$

etc., showing that the product of a positive number and a negative number is a negative number.

The children can also take reflection of these graphs in the x-axis to get multiplication table of (-2), (-3), (-4) etc. The dot-



ted line here gives graphs of multiplication table of (-2), its slope is (-2) and from the graph, we can read

$1\times(-2)=-2$, $2\times(-2)=-4$, $3\times(-2)=-6$, ...

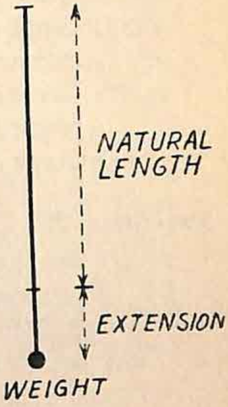
$(-1)\times(-2)=2$, $(-2)\times(-2)=4$, $(-3)\times(-2)=6$, ...

so that the product of a positive number and a negative number is negative and the product of two negative numbers is positive.

Experiment 398. Other Constant-Slope Graphs.

Constant-slope graphs represent direct proportionality or linear relationships which are of great importance. The children can draw the following graphs and notice that these are all constant-slope graphs.

- (*i*) Graph of circumference of a circle against its diameter : Give the children different circular discs and let them measure their circumferences by means of a thread and their diameters by using a scale. The graph will come out to be a straight line with slope slightly greater than 3. This slope of the constant ratio of circumference of a circle to its diameter is an important number in mathematics and is denoted by the Greek letter π. Tell the children that its approximate value in 22/7 or 3,14159, but for its exact value, we have to take infinite number of decimal places.
- (*ii*) Graph of the price of a number of copies of a book against the number of books (no bulk discount being assumed).
- (*iii*) Graph of the weight of a number of identical balls against the number of balls.
- (*iv*) Graph of the volume of a number of identical balls measured by means of a graduated cylinder, against the number of balls.
- (*v*) Graph of extensions of an elastic string beyond its natural length caused by different weights attached to its end against the weights.
- (*vi*) Graph of railway fares against distances. Different graphs will be obtained for air-conditioned, first, second and third classes.
- (*vii*) Graph of volume of water that falls from a tap in a graduated cylinder against the time.
- (*viii*) Graph of weights of different iron balls against their volumes.



(*ix*) Graph of weights of different volumes of water against their volumes.

(*x*) Graph of weights of different substances in ounces against their weights in grammes.

(*xi*) Graph of the height of first rebounce of an elastic ball against the height from which it is dropped.

(*xii*) Graph of the perimeter of a square against its side (compare it with the graph of multiplication table of 4).

(*xiii*) Graph of the wage of a worker against the number of days he works.

(*xiv*) Graph of the income-tax payable against the income (note that the slope changes as we go to different income-tax slabs).

(*xv*) Graph of the interest on a certain sum at a given rate against the time.

(*xvi*) Graph of the interest on different sums at a given rate for a given time against the sums.

(*xvii*) Graph of interest on a given sum for a given time against different rates of interest.

(*xviii*) Graph of cars produced by a factory against the number of weeks.

(*xix*) Graph of height of water in a cylinder from which water is leaking against the time.

(*xx*) Graph of number of words in a book against the number of pages (what do you notice?)

(*xxi*) Graph of the premiums to be paid at a certain age against the amount insured (you get different graphs for different ages).

(*xxii*) Graph of brokerage to be paid to a person against the amount of the work done by him.

(*xxiii*) Graph of distance travelled by a train against the time taken prepared with the help of a railway time table.

(*xxiv*) Graph of petrol used by a car against the distance travelled.

Experiment 399. Graphs to show how squares and cubes etc. grow.

(*i*) Plot the squares of the numbers 1, 2, 3, 4, 5,... against the numbers *i.e.*, plot the points (1, 1), (2, 4), (3, 9), (4, 16), (5, 25), ... and join them by a freehand curve.

(*ii*) Draw squares of sides 1, 2, 3, 4, 5, ... on graph paper, measure their areas and draw a graph of the areas of the squares against the lengths of their sides.

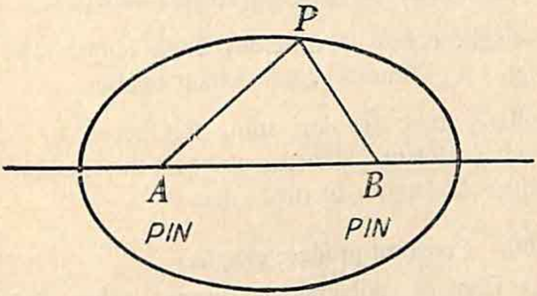
(*iii*) Draw equilateral triangles of sides 1″, 2″, 3″, 4″, 5″,... on a cardboard sheet. Cut these off, weigh them and draw the graph of their weights against the lengths of their sides.

(*iv*) Similarly draw circles of radii 1 cm., 2 cm., 3 cm., 4 cm., 5 cm.,... cut them off from card board sheets, weigh them and draw the graph of the weights against the radii or the diameters.

(*v*) Repeat the same experiment with regular hexagons, octagons etc.

(*vi*) Instead of drawing triangles, circles, hexagons, etc. on cardboard sheets, draw these on graph paper, measure their areas and draw graph of areas against the lengths of sides.

(*vii*) Draw a straight line AB 2″ long. Take a string 4″ long, attach it to two pins at A and B, place the pencil inside the string and draw an ellipse as shown below :



Cut a cardboard sheet of this size and weigh it. Draw other ellipses with (AB=3″, length of string 6″), (AB=4″, length of string 8″) and so on. Plot these weights against distance AB. The graph can also be drawn for the areas of the graphs against the distances between the focii A, B. [$A=\pi ab=\pi a^2\sqrt{1-e^2}$, $AB=2ae$, $AP+PB=2a$]

(*viii*) Draw rectangles with sides (2″, 1″), (4″, 2″), (6″, 3″), (8″, 4″) .. (*i.e.*, with constant ratio of sides) and plot their areas (or weights) against the lengths of the larger (or smaller) side.

(*ix*) Take a smooth plane of small inclination and roll a ball down the plane. Draw the graph of the distance travelled against the time taken. [$S=\frac{1}{2}g \sin \alpha\, t^2$]

(*x*) Take simple pendulums of different lengths. Find the time taken by each to beat 15 times and plot the length against time $\left[l=\frac{1}{4\pi}g\,t^2\right]$.

(*xi*) Suspend two slightly different masses over a smooth pulley and plot the distance travelled by one of the masses against the time taken $\left[\mathrm{S}=\frac{1}{2}\frac{m_1-m_2}{m_1+m_2}g\,t^2\right]$.

(*xii*) Take cubes of edges 1″, 2″, 3″, 4″, 5″,... Draw the graph of the area of a face against the length of an edge $[\mathrm{A}=6a^2]$. All the above graphs show the graph $y=kx^2$ and are parts of parabolic curves. These show how squares grow. The following graphs show how cubes grow :

(*xiii*) Take spherical balls of different diameters and plot their weights (or volumes) against their diameters.

(*xiv*) Take solid cubes of different edge length and plot their weights (or volumes) against their edges.

(*xv*) Similarly take regular solid tetrahedra or octahedra or icosahedra or dodacahedra and plot their volumes or weights against the lengths of their edges.

Experiment 400. Constant product graphs.

In most arithmetic problems, we discuss either direct variation or inverse variation. In the former case, the graph is a straight line represented by $y=kx$ and gives a constant slope graph. In the latter case the graph is a branch of a rectangular hyperbola and is represented by $y=\frac{k}{x}$. We call this graph as a constant-product graph, since here the product xy=constant.

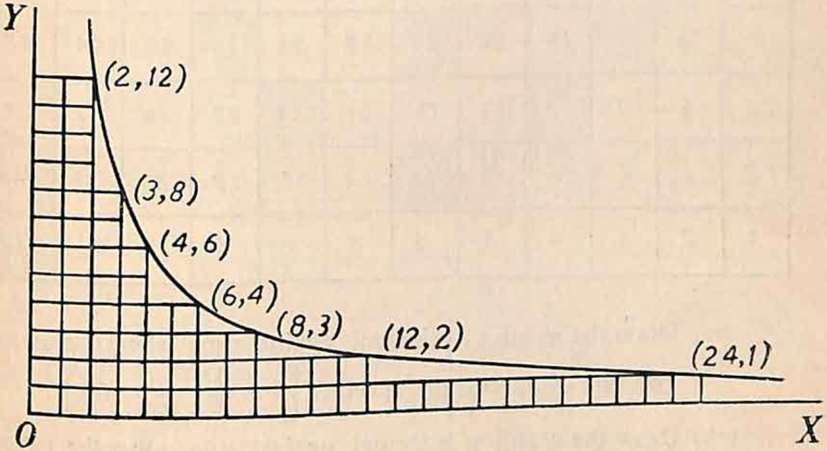
(*i*) Draw graph of the exterior angle of a regular polygon against the number of sides $\left[\text{exterior angle}=\frac{360^\circ}{n}\right.$, where n is the number of sides]. As the number of sides increases, the angle decreases. The angle can be made as small as we please by increasing the number of sides, but can never be zero. This is expressed by saying that as the number of sides becomes larger and larger, the angle becomes smaller and smaller. In technical language, we say that as n tends to infinity, the exterior angle tends to zero. What does the interior angle tend to ?

(*ii*) Draw graph of the angle subtended at the centre of a regular polygon against the number of sides

$$\left[\text{angle at the centre} = \frac{360^\circ}{n}\right].$$

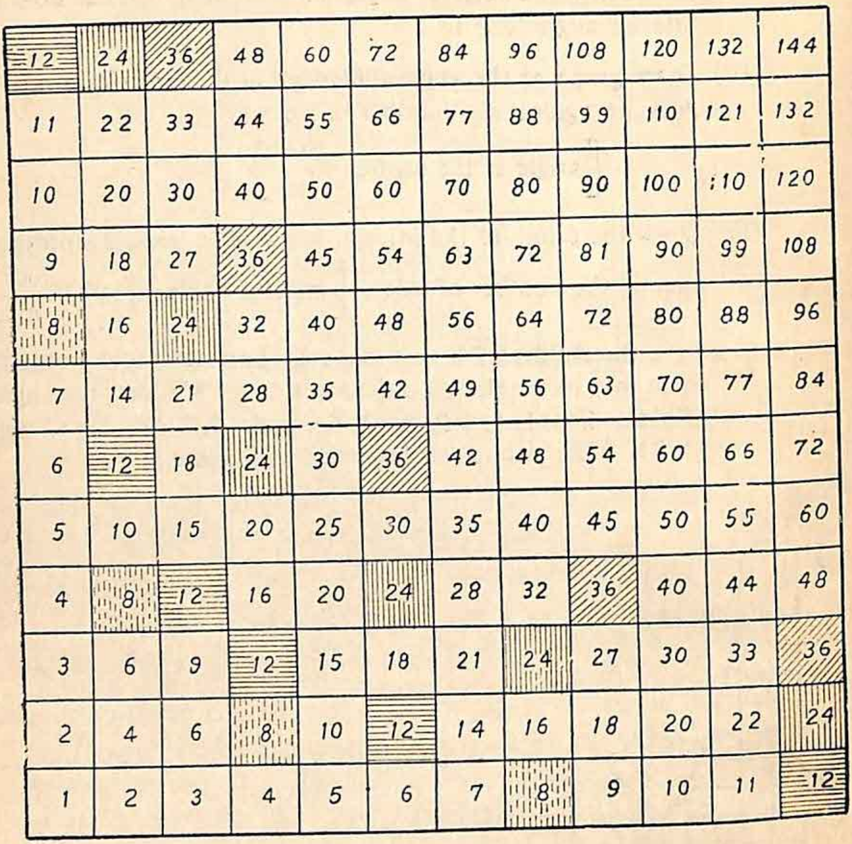
(*iii*) Draw the graph of the interior angle of a regular polygon against the number of sides. $\left[\text{interior angle} = 180^\circ - \frac{360^\circ}{n}\right]$.

(*iv*) Give the children 24 unit squares and let them combine these into rectangles of different sides. We get rectangles of sides 24×1, 12×2, 8×3, 6×4, 4×6, 3×8, 2×12 and 1×24. The points can be joined as follows :



Here we have rectangles of constant area and we have plotted the length of one side against the other side and the points lie on the rectangular hyperbola $xy=24$. The length of one side varies inversely as the length of the other side.

(*v*) Write the multiplication tables and shade all squares marked 8 or 12 or 24 or 36 and we get different hyperbolic arcs.



12	24	36	48	60	72	84	96	108	120	132	144
11	22	33	44	55	66	77	88	99	110	121	132
10	20	30	40	50	60	70	80	90	100	110	120
9	18	27	36	45	54	63	72	81	90	99	108
8	16	24	32	40	48	56	64	72	80	88	96
7	14	21	28	35	42	49	56	63	70	77	84
6	12	18	24	30	36	42	48	54	60	66	72
5	10	15	20	25	30	35	40	45	50	55	60
4	8	12	16	20	24	28	32	36	40	44	48
3	6	9	12	15	18	21	24	27	30	33	36
2	4	6	8	10	12	14	16	18	20	22	24
1	2	3	4	5	6	7	8	9	10	11	12

(*vi*) Draw the graph of principal against time when the interest and rate of interest are specified $\left[P=\frac{1}{T}\left(\frac{I}{R}\times 100\right)\right]$.

(*vii*) Draw the graph of principal against rate when the interest and time are specified $\left[P=\frac{1}{R}\left(\frac{I\times 100}{T}\right)\right]$.

(*viii*) Draw the graph of rate against time when the interest and principal are specified $\left[R=\frac{1}{T}\left(\frac{I\times 100}{P}\right)\right]$.

(*ix*) Suppose 36 men are required to finish work in one day. Draw the graph of the number of men required against the number of days in which work has to be finished [Number of men × number of days = 36].

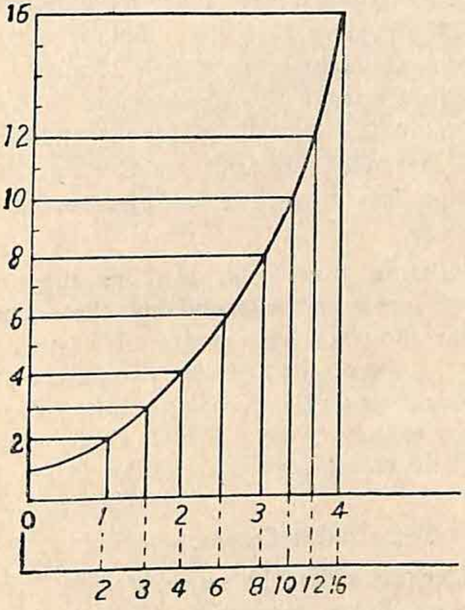
(x) Draw the graph of the velocity against time to cover a certain distance [V × T = S].

The students can be asked to think of other situations where one quantity increases, the other decreases and the product remains constant and to draw graphs for such situations.

Experiment 401. Growth of Powers of Two. Binary Slide Rule.

(*i*) Draw the graph of powers of 2 by joining the points (1, 2), (2, 4), (3, 8), (4, 16), (5, 32), (6, 64),...... Draw also the graphs of multiplication table of 2, of squares and cubes and note that the powers of 2 increase much more rapidly than multiples of 2 or squares or cubes of numbers.

(*ii*) We can use 'powers' graph for multiplying numbers as follows :



To multiply two numbers on the Y-scale, we add the corresponding numbers on the X-scale and then find the corresponding number on the Y-scale. This is based on the relation $2^a \times 2^b = 2^{a+b}$. Thus to multiply 2 and 8, we find that the corresponding numbers on the X-scale are 1 and 3, their sum is 4 and the corresponding number on the Y-scale is 16 which is the product of 2 and 8.

We can construct a Binary Slide Rule by using this graph. We renumber points in the X-scale and give them same numbers as on the Y-scale. The new scale can be transferred to two strips on drawing paper or wooden or plastic scales. When these are made to slide upon one another, the products and the quotients may be read directly. The children may like to make their own slide rules.

(*iii*) Recreational value of powers of 2.

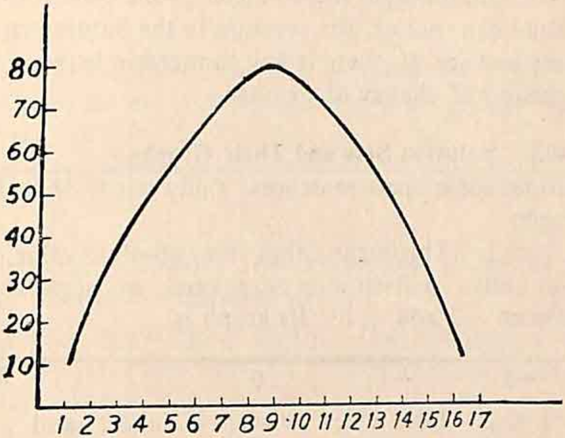
Children enjoy and appreciate the rapid growth of powers of 2 if some recreational examples are given.

(*a*) Take a paper $\frac{1}{128}''$ thick. Fold it once, its thickness becomes $\frac{1}{64}''$. Fold it once again, its thickness becomes $\frac{1}{32}''$. By folding it seven times, its thickness becomes 1″. By folding it seven times more its thickness becomes 128″ or more than 10′. Fold it seven times more its thickness becomes more than 32 miles. Another seven folds will make its thickness more than 4000 miles. A total of 42 folds will make the thickness more than 50,000 miles. If (and this is a big if) we can fold the paper fifty times, we can go beyond the planetory system.

(*b*) A's father gives to his son one rupee on the first day, two rupees on the second day, three rupees on the third day and so on for a month. B's father gives to his son one paisa on the first day, two paise on the second day, four paise on the third day and he goes on doubling the payment every day. Which child is richer at the end of the month ?

Experiment 402. Some Other Graphs.

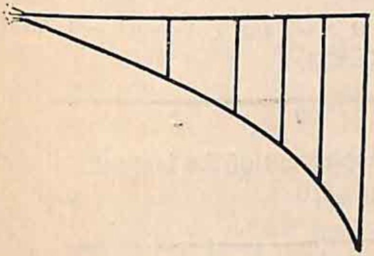
(*i*) Consider rectangles with same parameter and draw a graph of the area of the rectangle against a side *e.g.*, if the parameter is 36 cm and we take sides as 1 cm, 2 cm, 3 cm, ..., 9 cm, 10 cm, ..., 17 cm, we get the areas as 17, 32, 45, 56, ..., 80, 81, 80, ..., 17 square cm. The graph comes out as follows :

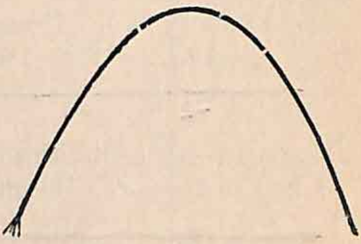


The graph shows a certain symmetry and the area is maximum when the side is 9 cm. *i.e.*, when the rectangle is a square.

This example shows how area increases and then decreases and reaches a certain maximum value.

(*ii*) Take a photograph of a water nozzle coming out of a horizontal spout and measure horizontal and vertical ditsances.





What relation do you find ? For an inclined spout the graph is parabolic.

(*iii*) The children may draw the graphs of the heights of some quick-growing plant in the school garden against its age in weeks. Similarly they can draw graph of weight of their new-born brother or sister week by week. The graphs will not be as smooth as those we have drawn earlier, but still will show useful patterns.

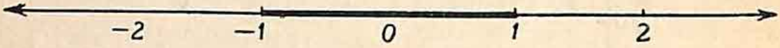
(*iv*) A child can draw two graphs of maximum and minimum temperature day by day as he notes the temperature in a maximum-minimum thermometer in the class room.

(*v*) A child can notice the pressure in the barometer every two hours and see if there is any connection between change of pressure and change of weather.

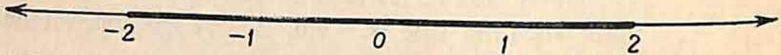
Experiment 403. Solution Sets and Their Graphs.

We consider some open sentences, find their truth sets and then draw their graphs.

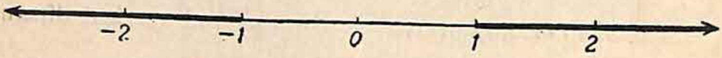
*(*i*) $|x| < 1$. This means that the absolute value of x is less than unity. x itself may be positive or negative and lies between -1 and $+1$. Its graph is



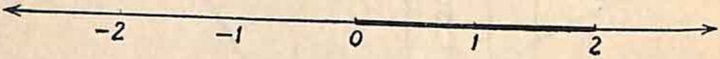
(*ii*) $|x| \leqslant 2$. Here x lies between -2 and 2 and x can also be equal to -2 and $+2$, so that its graph is



(*iii*) $|x| > 1$. Here x can be positive and greater than 1 or x can be negative and less than 1. x is not equal to -1 or $+1$. Its graph is

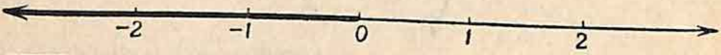


(*iv*) $|x-1| \leqslant 1$. This means that $x-1$ lies between -1 and $+1$ *i.e.*, $-1 \leqslant x-1 \leqslant 1$ or $0 \leqslant x \leqslant 2$.



(*v*) $|x-1| > 1$. This is the complement of the last set.

(*vi*) $|x| = -x$. This means that $x \leqslant 0$.



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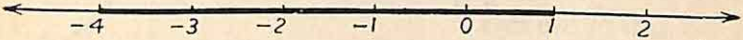
*Note : In (*i*) the truth set *includes all* real numbers lying between -1 and 1 and *excludes* the numbers -1 and 1. Thus the number ·999 ... (even to hundred places of decimal) is a member of the truth set, though ·999 ... (to an infinite place of decimal) is not (why ?). Geometrically it is not possible to draw a precise graph. However, it must be understood in the above graphs that the radii of the circles can be made as small as we please.

(*vii*) $|x| = -4$. The solution is given by the null set.

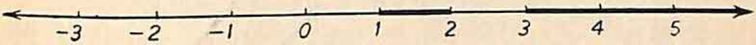
(*viii*) $\left|\frac{x}{x}\right| = \frac{x}{x}$. This is satisfied by all values of $x>0$ and all values of $x<0$ but not by $x=0$.

(*ix*) $\{x \mid x<1 \wedge x>-4\}$. This means the set of all these values of x which are less than unity and are greater than -4.

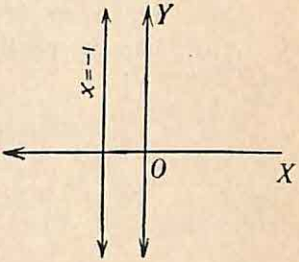


(*x*) $\{x \mid |x| > -4\}$. This is true for all values of x. The truth set is the universal set.

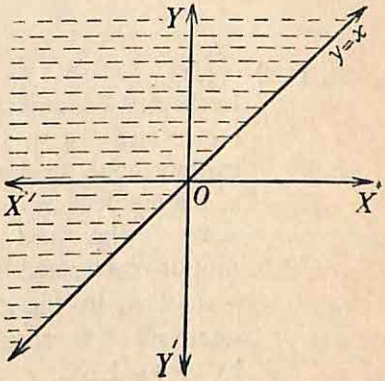
(*xi*) $\{x \mid (x-1)(x-2)(x-3)>0\}$. The sentence $(x-1)(x-2)(x-3)>0$ is true when $1<x<2$ and when $x>3$ so that the graph is



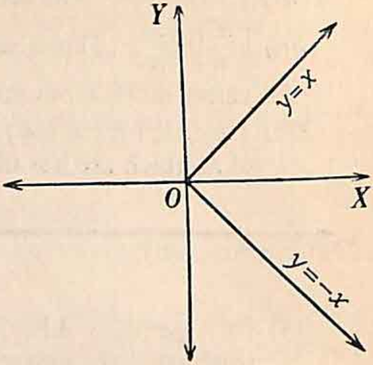
(*xii*) $\{(x, y) \mid x+1=0\}$. The sentence is true for all ordered pairs of which the first member is -1. In the Cartesian plane, it represents the straight line $x=-1$.



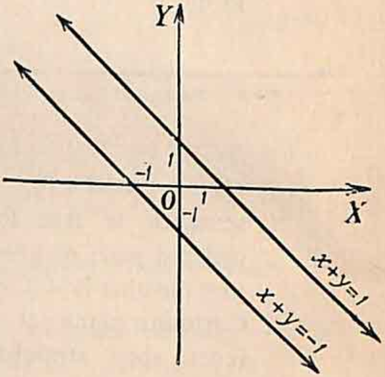
(*xiii*) $\{(x, y) \mid y>x\}$. The graph consists of all points in the Cartesian plane of which the y-coordinate is greater than the x-coordinate.



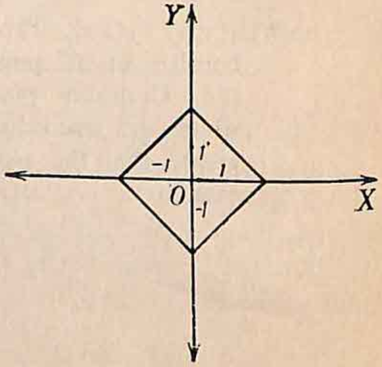
(*xiv*) $\{(x,\ y) \mid |\,y\,| =x\}$. If y is positive, we get $x=y$ and if y is negative, we get $x=-y$. The graph consists of two rays.



(*xv*) $\{(x, y) \mid |\,x+y\,| =1\}$. This represents two straight lines $x+y=1$ and $x+y=-1$.



(*xvi*) $\{(x, y) \mid |\,x\,| + |\,y\,| =1\}$. In the first quadrant, it is represented by $x+y=1$. In the second quadrant, it is represented by $-x+y=1$. In the third quadrant, it represented by $-x-y=1$. In the fourth quadrant it is represented by $x-y=1$.

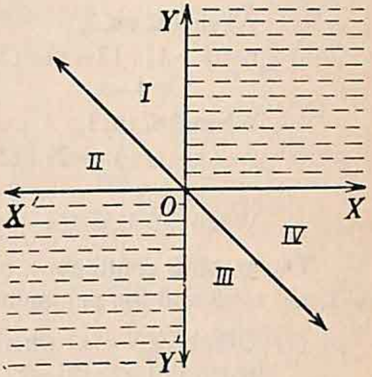


(*xvii*) $\{(x, y) \mid x+y=2 \wedge x-y=0\}$. The truth set consists of only one point *viz.*, (1, 1).

(*xviii*) $\{(x, y) \mid x+y=2 \wedge x+y=3\}$. The truth set is the null set.

(*xix*) $\{(x, y) \mid x+y=2 \wedge 2x+2y=4\}$. The truth set is the set of all points on the line $x+y=2$.

(*xx*) $\{(x, y) \mid |x+y| = |x| + |y|\}$. In the first quadrant $x+y=x+y$ so that all points are included. In the second quadrant region I $x+y=-x+y$ or $x=0$. In the second quadrant region II $-(x+y)=-x+y$ or $y=0$. In the third quadrant $-(x+y)=-x-y$ so that all points are included. In the fourth quadrant region III $-(x+y)=x-y$ so shat $x=0$. In the fourth quadrant region IV $x+y=x-y$ so that $y=0$.

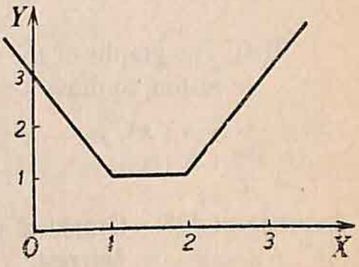


Experiment 404. Graphs of Some Special Functions.

When to every value of x in some range, we get a unique value of y, we say we are given a function and we write $y=f(x)$. To draw the graph, we give different values of x and find the corresponding values of y.

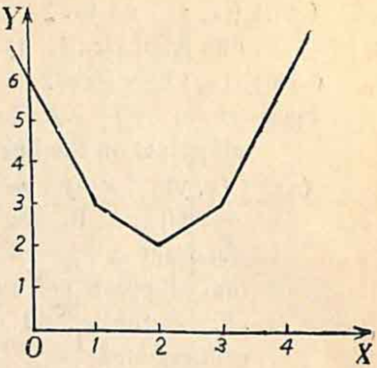
(*a*) Graph of $y = |x-1| + |x-2|$

When $x \leqslant 1$,
$y=(1-x)+(2-x)=3-2x$
When $1 \leqslant x \leqslant 2$,
$y=(x-1)+(2-x)=1$
When $x \geqslant 2$,
$y=(x-1)+(x-2)=2x-3$.



The graph is continuous but its slope is discontinuous at $x=1$ and at $x=2$ and the minimum value of y occurs when x lies between 1 and 2.

(*b*) Graph of y
$= |x-1| + |x-2| + |x-3|$
When $x \leqslant 1$,
$y=(1-x)+(2-x)+(3-x)$
$=6-3x$
When $1 \leqslant x \leqslant 2$,
$y=(x-1)+(2-x)+(3-x)$
$=4-x$
When $2 \leqslant x \leqslant 3$,
$y=(x-1)+(x-2)+(3-x)$
$=x$

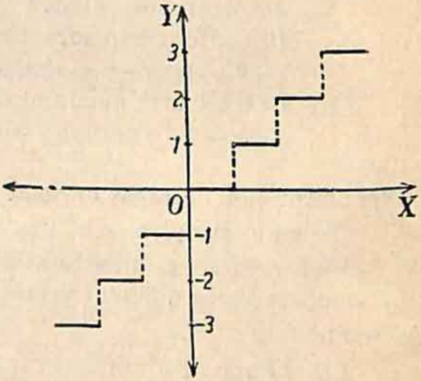


When $x \geqslant 3$, $y=(x-1)+(x-2)+(x-3)=3x-6$

The graph is continuous, but its slope is discontinuous at $x=1$, $x=2$ and $x=3$ and the minimum value occurs at $x=2$.

(*c*) Graph of $y=[x]$ where $[x]$ denotes the integral part of x so that $[3{\cdot}5]=3$, $[2]=2$, $[-1{\cdot}5]=-2$

When $-2 \leqslant x < -1$,
$y=-2$
When $-1 \leqslant x < 0$, $y=-1$
When $0 \leqslant x < 1$, $y=0$
When $1 \leqslant x < 2$, $y=1$
The graph is discontinuous for all integral values of x.



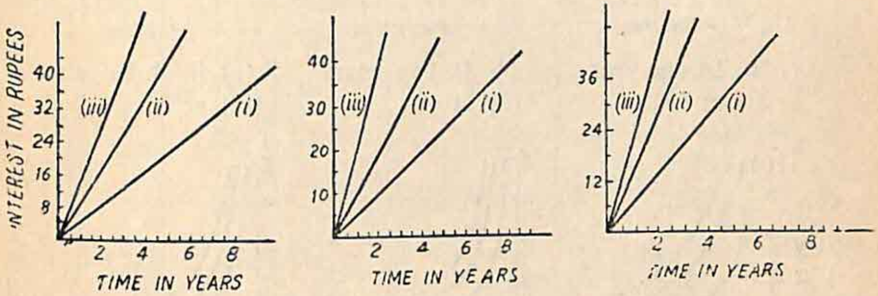
(*d*) The graphs of the following functions would be quite interesting to draw

$y=|x|$, $y=-|x|$, $y=|x-1|+|x-2|+|x-3|+|x-4|$
$y=|x|+[x]$, $y=-[x]$, $y=|[x]|$

Experiment 405. Graphical Representation for Problems of Simple Interest.

Most of the ordinary school arithmetic is concerned with the relationships $y=kx$ and $y=\frac{k}{x}$. Graphical methods can be used to impress visually, on the minds of the students, the significance of the direct proportion and inverse proportion relationships. We illustrate this by considering the case of simple interest.

Let the children draw the following graphs :—

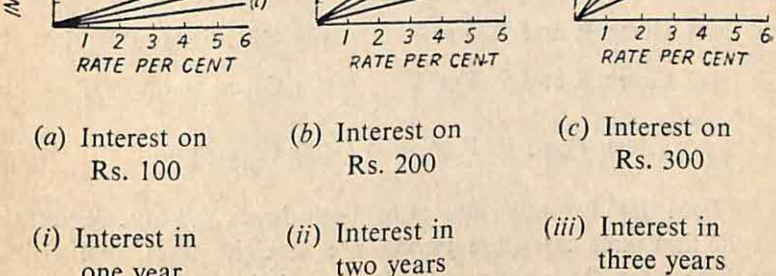


(*a*) Rate of interest =4%

(*b*) Rate of interest =5%

(*c*) Rate of interest =6%

(*i*) Interest on Rs 100 (*ii*) Interest on Rs 200 (*iii*) Interest on Rs 300



(*a*) Interest on Rs. 100

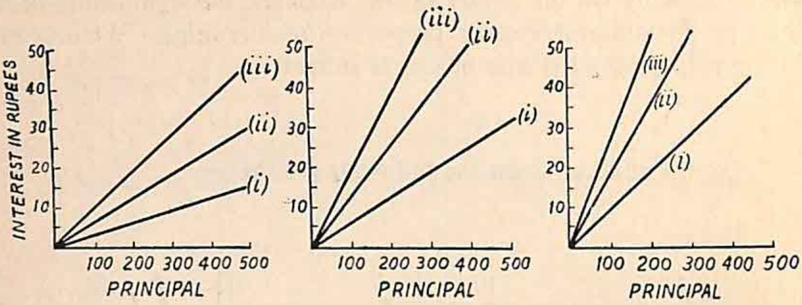
(*b*) Interest on Rs. 200

(*c*) Interest on Rs. 300

(*i*) Interest in one year

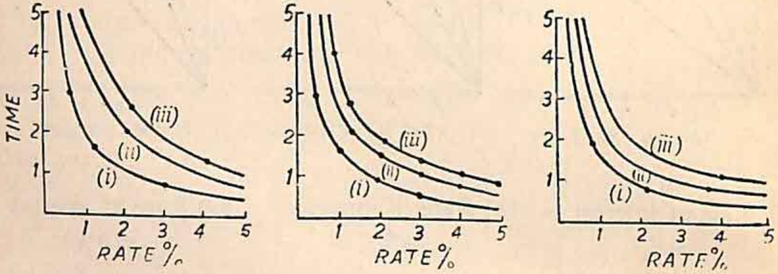
(*ii*) Interest in two years

(*iii*) Interest in three years



(a) In one year (b) In two years (c) In three years

(i) at 3% (ii) at 4% (iii) at 5%



(a) Principal =Rs. 100 (b) Principal =Rs. 200 (c) Principal =Rs. 300

(i) Interest=Rs. 3 (ii) Interest=Rs. 4 (iii) Interest=Rs. 5

The formula for simple interest is

$$I = \frac{P \times R \times T}{100},$$

where I is the interest in Rs., P is principal in Rs., R is rate of interest per hundred rupees per year in Rs. and T is the time in years. This shows that

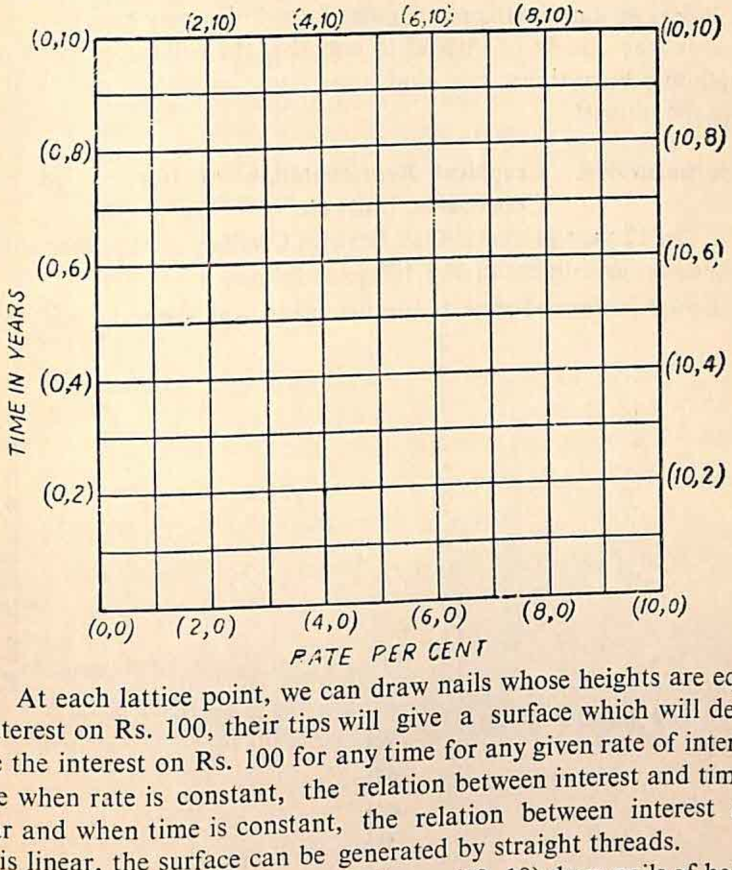
(i) Given R and T, $I \propto P$ (ii) Given P and T, $I \propto R$

(iii) Given R and P, $I \propto T$ (iv) Given I and P, $R \propto \frac{1}{T}$

(v) Given I and R, $P \propto \frac{1}{T}$ (vi) Given I and T, $P \propto \frac{1}{R}$.

Thus this formula illustrates both direct and inverse variation. For the first three cases, the graphs are straight lines, while for the last three cases, these are rectangular hyperbolas.

All these formulae can be easily illustrated in a rectangular hyperbolic paraboloid which can be constructed very easily by children.



At each lattice point, we can draw nails whose heights are equal to interest on Rs. 100, their tips will give a surface which will determine the interest on Rs. 100 for any time for any given rate of interest. Since when rate is constant, the relation between interest and time is linear and when time is constant, the relation between interest and rate is linear, the surface can be generated by straight threads.

At points (10, 1), (10, 2), (10, 3), ..., (10, 10) draw nails of height 10, 20, 30, 40, ..., 100 and join their tips by straight threads to the points (0, 1), (0, 2), (0, 3), ..., (0, 10). Similarly at points (1, 10), (2, 10), (3, 10), ..., (10, 10), draw nails of heights 10, 20, ..., 100 and join their tips by tight threads to (1, 0), (2, 0), ..., (10, 0). The threads will all lie on a 'ruled' surface (*i.e.*, a surface generated by straight lines) which may be called the 'interest surface'. The surface is a rectangular hyperbolic paraboloid. Its section by vertical planes parallel to OX, OY are straight lines and its sections by horizontal planes are rectangular hyperbolas.

The children can also make other 'interest surfaces' when the principal is Rs. 200 or Rs. 300 by using threads of other colours.

If we place horizontal glass sheets perforated at all the lattice points at heights of 10, 20, ..., 100 and let the threads pass through the holes, we can see the rectangular hyperbolas very easily.

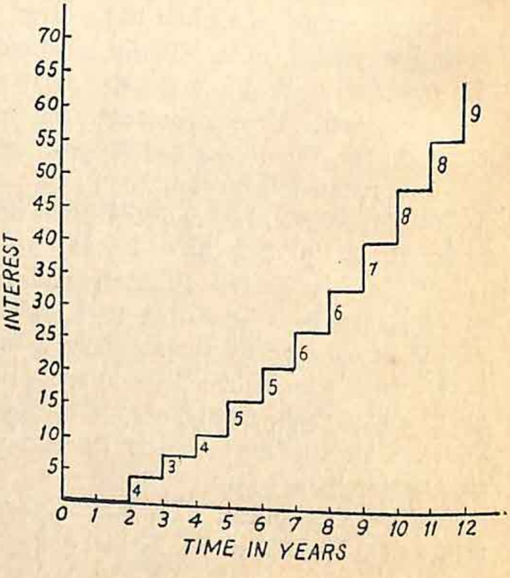
It may also be of interest to note that the volume of a box whose length and breadth are rate and time and height is principal / 100 gives the interest.

Experiment 406. Graphical Representation of Interest on Savings Certificates, Units etc.

On 12 year National Plan Savings Certificates, the amount to be claimed on investment of Rs. 100 is as follows :

If amount is claimed after	1	complete year from date of issue :	Rs. 100
,, ,, ,,	2	,, ,, ,,	Rs. 104
,, ,, ,,	3	,, ,, ,,	Rs. 107
,, ,, ,,	4	,, ,, ,,	Rs. 111
,, ,, ,,	5	,, ,, ,,	Rs. 116
,, ,, ,,	6	,, ,, ,,	Rs. 121
,, ,, ,,	7	,, ,, ,,	Rs. 127
,, ,, ,,	8	,, ,, ,,	Rs. 133
,, ,, ,,	9	,, ,, ,,	Rs. 140
,, ,, ,,	10	,, ,, ,,	Rs. 148
,, ,, ,,	11	,, ,, ,,	Rs. 156
,, ,, ,,	12	,, ,, ,,	Rs. 165

The graph is given below:



On 12-year Post Office National Savings Certificates, the successive amounts to be received at the ends of successive years are Rs. 100, 100, 105, 110, 115, 120, 125, 130, 135, 140, 145, 150. The children can draw its step-function graph.

On 12-year National Defence Certificates, the successive amounts to be received at the ends of successive years are Rs. 100, 104, 109, 114, 120, 126, 133, 140, 148, 156, 165, 175. The children can again draw its step-function graph.

On 10-year National Saving Certificates, the successive amounts to be received at the ends of successive years are Rs. 100, 108·50, 113·50, 119, 126, 135, 145, 156, 168, 180. The children can again draw a step-function graph.

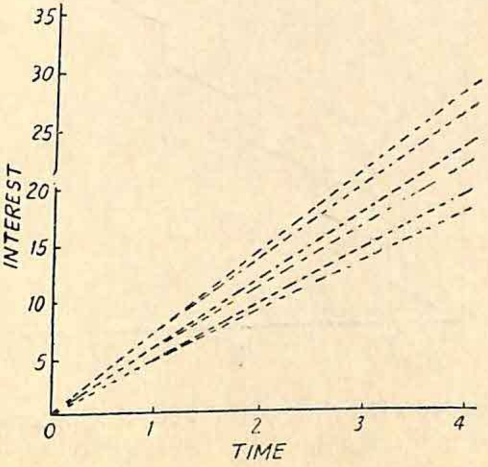
The children may also draw all these graphs in a single figure and get a comparative estimate of yields. They can also draw linear graphs of 4 percent, 5 percent, 6 percent and compare their yields.

The children may also find purchase and sale prices of units throughout the year and obtain their graphs.

These step-function graphs are of great interest in higher mathematics.

Experiment 407. Compound Interest.

Let the children draw graphs of both simple and compound interests on Rs. 100 at different rates percent for different periods.



Simple interest graphs are straight lines, while compound interest graphs are curved lines

$$S.I. = R \times T$$

$$C.I. = 100\left(1+\frac{R}{100}\right)^T - 100$$

From these graphs we can solve many problems.

(*i*) If P, R, T are known, we can find interest by multiplying the ordinate by $\frac{P}{100}$.

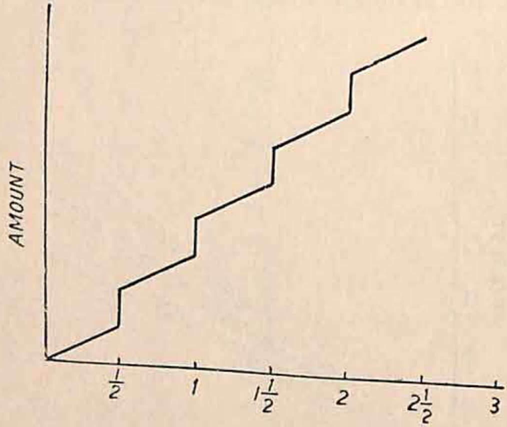
(*ii*) If I, P, R are known, we can find T by measuring $\frac{100}{P} \times I$ along interest axis and finding the corresponding abscissa.

(*iii*) If I, P and T are known, we can find R in the same way.

(*iv*) If difference between simple interest and compound interest for a given time and given rate is known, we can find the principal.

Those who know logarithms will find it more convenient to plot logarithm of amount/principal, against time. The graphs will be straight lines.

In banks where interest is paid half yearly, the graph will be as follows. Since during each six month paid, the interest is simple and



the interest is added after six months and the interest is paid on the total amount at that time.

Experiment 408. Graphical Representation of Other Problems of Arithmetic.

(*i*) **Time and Work** ; Work done=No. of persons × days × work done per man per day.

i.e., $W = N \times D \times R$

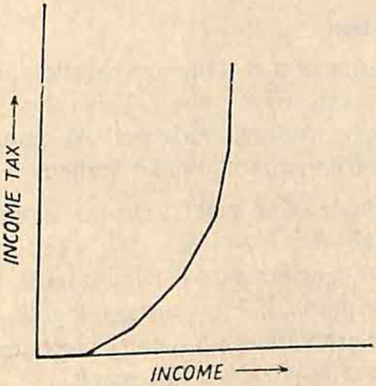
This is similar to the formula for simple interest and we find

(*i*) If D and R are fixed, $W \propto N$.
(*ii*) If N and R are fixed, $W \propto D$.
(*iii*) If N and D are fixed, $W \propto R$.
(*iv*) If W and N are fixed, $R \propto \frac{1}{D}$.
(*v*) If W and D are fixed, $N \propto \frac{1}{R}$.
(*vi*) If W and R are fixed, $N \propto \frac{1}{D}$.

In the first three cases, we get straight line graphs and in the last three cases, we get rectangular hyperbolas.

All the graphs can be represented on a 'work surface' which will be a rectangular hyperbolic paraboloid.

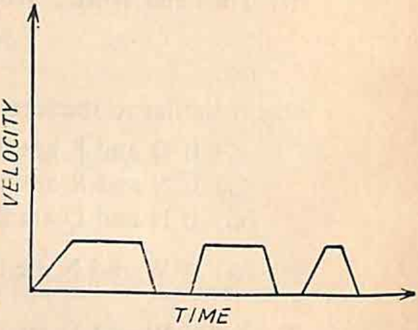
(*ii*) **Income Tax :** Since income tax rate varies in different slabs of income, the graph will be as follows :



(*iii*) **Commission, Brokerage, Sales Tax etc. :** Here the graphs are usually straight lines.

(*iv*) **Time and Motion** : The velocity-time graph will usually be as described below :

As a train leaves a station, it accelerates, its velocity increases, then for some time its velocity remains constant and then it decreases as the next station comes near. There the velocity remains zero for some time and then it changes in the same manner again till the next station. The area under the velocity-time graph will determine the distance covered.



From the railway time-table, the children may find average velocities between different stations for important trains, the average velocity for the whole journey and (knowing the time required for acceleration and retardation), the constant velocities between different stations. They may also find how much reduction in time will take place if the velocities are increased in a certain ratio.

Many more realistic problems in this field can be done than are done at present.

(*v*) **Mensuration** :

(*a*) Volume of a rectangular parallelopiped of edges a, b, c is given by $V=abc$. This gives rise to straight line graphs, rectangular hyperbola graphs and rectangular hyperbolic paraboloidal surfaces.

(*b*) Volume of a right circular cylinder of radius r and height h is $V=\pi r^2 h$. If h is constant, $V \propto r^2$ giving a parabolic graph. If r is constant, $V \propto h$ giving a straight line graph, if V is constant it gives $r^2 h$=constant so that the graph is similar to the rectangular hyperbola. Let the children draw this graph.

(*c*) Volume of a right circular cone of radius r and height h is $V=\frac{1}{3}\pi r^2 h$. The graphs are similar to those for the case of a cylinder.

(*d*) Curved surface area of a right circular cylinder is $S=2\pi rh$. Here r=constant gives a straight line, h= constant gives a straight line while S=constant giving a rectangular hyperbola.

(*e*) Curved surface area of a right circular cone is given by $S=\pi r\sqrt{h^2+r^2}$ so that r=constant gives a hyperbola while h=constant and S=constant give special types of curve which the children may like to draw.

(*f*) Surface area of the four walls of a room is given by $S=2(l+b)h$. If l, b are constant, we get a straight line, if l and h are constant, we get a straight line not passing through the origin, if b and h are constant, we get a straight line not passing through the origin, if S and h are constant, we again get a straight line. If S and l or S and b are constant we get rectangular hyperbolas.

Children may similarly draw graphs to represent other formulae of mensuration.

12

Congruence

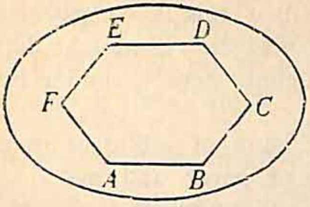
Experiments : 409 to 417

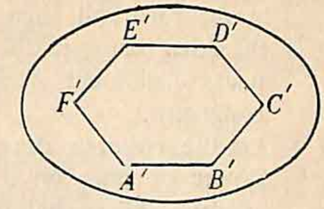
Experiment 409. The Study of Cogruence.

Congruence means identity of shape and size. Recognition of identity of shape and size of objects is an important mathematical concept. If objects have same shape, but different sizes, they are similar but not congruent. Children can be shown different objects and asked to find which are congruent, which are similar but not congruent and which are neither congruent nor similar. The objects can include spheres of different radii, right circular cones of different radii and heights, right circular cylinders of different shapes and sizes of bases and of different heights, toys of different sizes and shapes, flowers and fruits from same trees, circular discs of different radii, triangular discs of different shapes and sizes, polygonal regions of different shapes and sizes, elliptic discs, letters of the alphabet in different sizes, figures drawn on playing cards, discs of different shapes and sizes etc. Congruence and similarity for both two and three dimensional objects should be studied.

Experiment 410. Congruence of Two-Dimensional Objects.

For two dimensional or plane bodies or discs, the children can find whether two objects are congruent to each other by placing one upon the other *i.e.* two objects are congruent if they can be placed one upon the other so that they completely fit each other. By placing one object upon another, we establish a one-to-one correspondence between points of the one and points of the other such that the distance between any two points of one is equal to the distance between corresponding points of the other so that *e.g.* AB=A′B′, BC=B′C′, CD=C′D′, DE=D′E′,



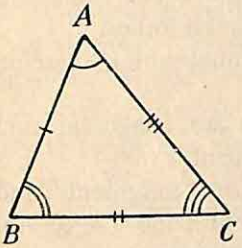


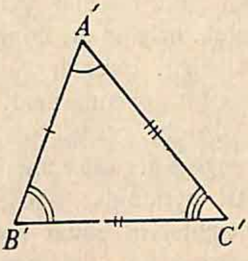
Also $\angle ABC=\angle A'B'C'$, $\angle AFE=\angle A'F'E'$ etc.

Let the children deduce experimentally that

(*a*) Two circular discs are congruent if and only if their radii are equal.

(*b*) Two triangles are congruent if three sides of one are equal to three sides of the other, each to each and the three angles of the one are equal to the three corresponding angles of the other.





It can be shown that the following conditions are sufficient for congruence of triangles

(*i*) $AB=A'B'$, $BC=B'C'$, $AC=A'C'$

(*ii*) $AB=A'B'$, $BC=B'C'$, $\angle ABC=\angle A'B'C'$

(*iii*) $BC=B'C'$, $\angle ABC=\angle A'B'C'$, $\angle ACB=\angle A'C'B$

However the equality of three angles is not sufficient, for in that case the triangles can be similar without being congruent.

(*c*) Two polygons are congruent if

(*i*) The number of sides is the same.

(*ii*) We can establish a correspondence so that the corresponding sides are equal and the corresponding angles are equal.

(*d*) The figures of hearts (or spades or diamonds or clubs) in a given pack of cards are congruent. To verify this, the children can cut out a paper figure congruent to a figure in one card and then check that this is congruent to all the other figures in the pack of cards. If we take different packs of different sizes, the figures may be similar but not congruent.

(*e*) Let the children draw any figure of a bird or animal or flower or man on a piece of paper and make carbon copies. All carbon figures are congruent to the original figures.

(*f*) In the same way, let the children take 6 or 10 folds of paper, draw a figure and cut out this figure on all the folds with a pair of scissors. All the figures are congruent.

(*g*) Regions bounded by congruent figures are also congruent.

Experiment 411. Congruence in Space and Congruence of Angles. The children may convince themselves that

(*a*) All spheres of equal radii are congruent to one another.

(*b*) All right circular cylinders with equal radii of base and equal heights are congruent to one another.

(*c*) All right circular cones with equal radii of base and equal heights are congruent.

(*d*) All right prisms whose bases are congruent and whose heights are the same are congruent.

(*e*) All pyramids whose bases are congruent and whose heights are equal are congruent if the projections of the vertices on the bases are corresponding points in the congruence mapping.

To establish congruence for three-dimensional bodies in space is not so easy as to establish congruence for two-dimensional bodies. A two-dimensional body can be 'lifted' up in the third dimension and put on the other body to test whether the second body is congruent to the first. A three-dimensional body will have to be 'lifted' in a fourth dimension which is not physically realisable.

Another method is to see whether the first body can occupy the position in space at present occupied by the second body by (*i*) translation of the body in some direction through some specified distance (*ii*) rotation of the first body about some axis (*iii*) reflections in some planes or (*iv*) combination of these.

Alternatively we may try to make moulds of one body and try to fit these on others.

An angle is the union of two rays and two angles will be congruent if by reflection, rotation, translation, glide reflection and combination of these, we can make the two rays of one angle coincide with two rays of the other.

Experiment 412. Congruence of Triangles and Polygons.

We have discussed above the congruence of two triangles. Polygons can be broken up into triangles and as such congruence of polygons can be studied in terms of those of triangles.

It is obvious that if all the sides of one polygon of more than three sides are equal to the corresponding sides of another polygon, then the polygons need not be congruent. The children can easily demonstrate this by taking 4 or 5 or 6 rods of specified lengths joined by pins to form a polygon. The frame is not rigid and can be deformed into other polygons with different angles. For triangles, this will not be possible.

It is interesting to find the conditions for congruence which are short of requiring all sides and all angles to be equal *e.g.* we may require all sides and all angles except one to be equal. Two polygons will also be congruent if not only lengths of all sides, but lengths of all diagonals from all vertices (or even one vertex) are equal, for in that case all triangles formed by the sides and the diagonals will be congruent.

Experiment 413. The Nature of the Congruence Relation.

Congruence can be regarded as a one-one onto mapping from one body to another body which preserves distances and angles.

Congruence is also a relation which is an equivalence relation since

(*i*) Every body is congruent to itself (reflexive property)

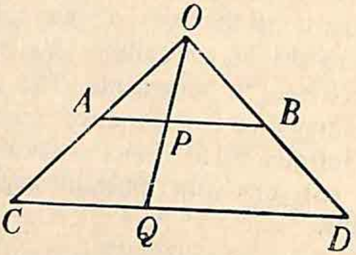
(*ii*) If body A is congruent to body B, then body B is congruent to body A (symmetric property)

(*iii*) If a body A is congruent to body B and body B is congruent to body C, then body A is congruent to body C (transitive property).

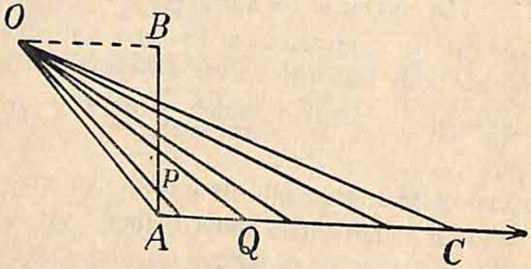
Thus all bodies can be divided into disjoint equivalence classes such that bodies within each class are congruent to one another while nobody of one class can be congruent to a body of another class.

It may be noted that congruence is a mapping which (*i*) preserves distances (*ii*) preserves angles (*iii*) preserves areas and volumes (*iv*) does not preserve location in space (*v*) does not preserve mass or density or temperature or colour etc.

It may also be noted that every one-one mapping is not a congruence mapping *e.g.* the projective mapping of line segment $\overline{AB}$ to segment $\overline{CD}$ is one-one, since to every point P of $\overline{AB}$ there corresponds a unique point Q of $\overline{CD}$ and to every point Q of $\overline{CD}$ there corresponds a unique point P of $\overline{AB}$, yet it is not a congruence mapping, since it does not preserve lengths.



In fact the projective mapping (shown below) of the line segment $\overline{AB}$ to the ray $\overrightarrow{AC}$ is one-one, yet the length $\overline{AB}$ is finite while the length of $\overrightarrow{AC}$ is infinite.

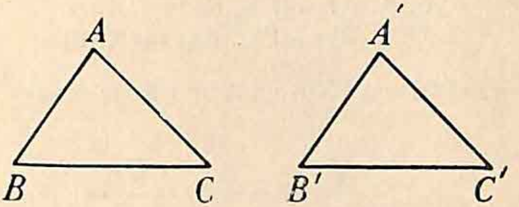


Experiment 414. Group of Congruence Mapping.

In our discussion of congruence mappings above, we have wanted both distances and angles to be preserved. We can however see easily that if all distances are preserved, then all angles must be preserved.

Let A, B, C be three non-collinear points which are mapped onto non-collinear points A′, B′, C′ by a distance-preserving mapping, then

AB=A′B′, BC=B′C′, CA=C′A′, so that triangles ABC and A′B′C′ are congruent and therefore the three angles of one triangle are equal to three angles of the other, each to each. Thus all angles are also preserved.



Let F denote the congruence mapping so that FA, FB denote images of points A and B. Also let $d(A, B)$ denote the distance between A and B, then F will be a congruence mapping if for all points A, B

$$d(FA, FB)=d(A, B)$$

(*i*) Let F_1, F_2 denote two congruence mappings, then

$$d(F_2F_1A, F_2F_1B)=d(F_1A, F_1B)=d(A, B)$$

so that $F_2 F_1$ is also a congruence mapping.

(*ii*) Since F_1, F_2, F_3 are one-one onto mappings

$$F_1(F_2F_3)=(F_1F_2)F_3,$$

so that composition of congruence mappings is associative.

(*iii*) There exists an identity congruence mapping I such that

$$IA=A \text{ for all points } A$$

and this mapping is such that

$$FI=IF=F$$

(*iv*) To every one-one onto mapping F, there exists an inverse mapping F^{-1} such that $FF^{-1}=F^{-1}F=I$. Also

$$d(FF^{-1}A, FF^{-1}B)=d(IA, IB)=d(A, B)$$
$$d[F(F^{-1}A), F(F^{-1}B)]=d(F^{-1}A, F^{-1}B)=d(A, B)$$

so that F^{-1} is also a distance preserving one-one mapping and therefore is a congruence mapping.

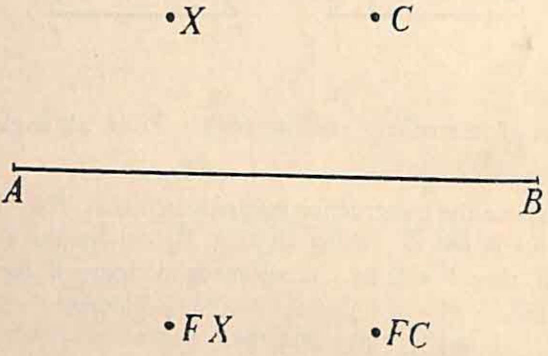
Thus congruence mappings form a group.

Experiment 415. Congruence Mappings Which Leave Two Points Fixed.

Let F be such a mapping, then FA=A, FB=B. Also for any point X

$$d(FX, A)=d(FX, FA)=d(X, A)$$
$$d(FX, B)=d(FX, FB)=d(X, B)$$

so that distances of X and FX from A and B are equal.



so that either FX=X or FX is image for X in the line AB.

If FX=X for all X, the congruence mapping is identity mapping I. If FX≠X for all X, there is a point *C* for which FC≠C. Let X be a point such that FX=X, then

$$d(X, C)=d(FX, FC)=d(X, FC)$$

so that X lies on the perpendicular bisector of the line joining C and FC *i.e.*, X lies on the line AB. The points X, not on the line AB, thus do not remain fixed and for them FX≠X. Thus FX is the mirror image of X in the line AB. This holds good for points on AB also. Thus we have proved that a *congruence mapping which leaves two distinct points A, B fixed is either I or is the reflection in the line AB.*

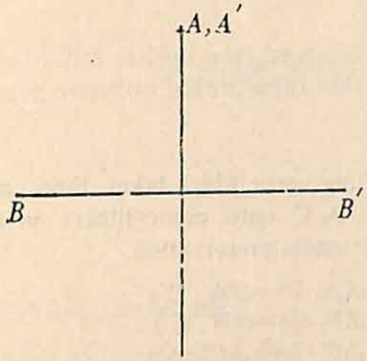
Experiment 416. Congruences Which Take Points A, B into A′, B′ Where d(A, B)=d(A′, B′).

We want a congruence F which is such that

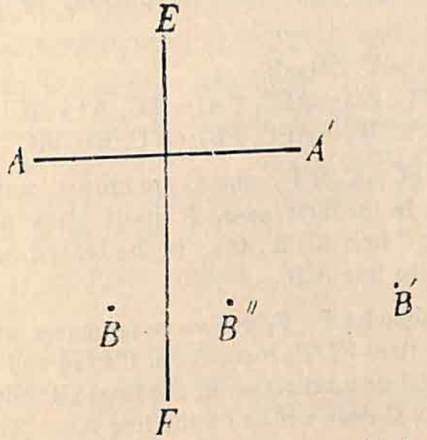
$$FA=A', FB=B'$$

Case I : A=A′, B=B′, then I is the required congruence.

Case II : A=A′, B≠B′, then A (A′) lies on the right bisector of BB′ and reflection in this line gives the required congruence.



Case III : A≠A′, B≠B′.



We first take a reflection R in the perpendicular bisector of AA′. This takes A into A′ and B into B″ such that

$d(A, B)=d(A', B'')=d(A', B')$

then a further rotation about point A′ will bring point B″ to B′ and this rotation will leave A′ unchanged. Thus a reflection in EF followed by a rotation about A′ (both of which are congruences) will transfer A to A′ and B to B′.

If we further take a reflection in the line A′B′, these points remain unchanged.

Thus we have found two congruences which take A to A′, B to B′ *viz.*,

(*i*) reflection in the right bisector of AA′ followed by rotation about A′

(*ii*) the above followed by a further reflection in A′B′. It can be shown that these are the only two possible congruences of this type.

Experiment 417. Congruence which takes three non-collinear points A, B, C onto non-collinear points A′, B′, C′ with distances preservation.

We are given $d(A, B)=d(A', B')$
$d(B, C)=d(B', C')$
$d(C, A)=d(C', A')$

In the last experiment, we have found two congruence mappings which take A, B onto A′, B′. Let F be one-one of these mappings, then

$$FA=A', FB=B'$$
$$d(FC, A')=d(FC, FA)=d(C, A)=d(C', A')$$
$$d(FC, B')=d(FC, FB)=d(C, B)=d(C', B')$$

Thus either FC=C or FC and C are mirror images of each other in the line A′B′. In the first case, F itself gives a transformation which takes A, B, C into A′, B′, C′. In the second case we have to take reflection in the line A′B′.

Now if possible let F_1, F_2 be two congruences which take A, B, C onto A′, B′, C′, then $F_1^{-1}F_2$ leave A, B, C fixed and from experiment 415 must either be I or a reflection in the line AB, but the second case is impossible, since C does not lie on the line AB. Thus

$$F_1^{-1}F_2 \text{ is } I \text{ or } F_1=F_2$$

Thus there is only one congruence map which takes A, B, C into A′, B′, C′ and at the same time preserves distances.

13

Reflection Geometry

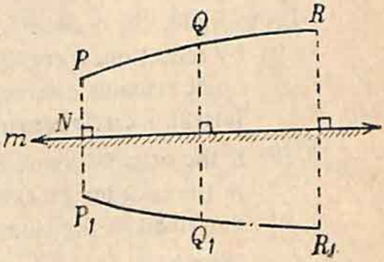
Experiments : 418 to 429

Experiment 418. Reflection in a Line.

One of the objects of the study of geometry is to study those properties of geometrical figures which do not change under simple transformations like reflections, translations, rotations and their combinations. The study of these transformations also enables us to study the symmetries exhibited in painting, sculpture and nature.

We start with the study of reflections.

A geometric object is a set of points. By reflection in a mirror, a point P goes to its image which is another point P_1. If from P, we draw a perpendicular PN to the mirror and produce it to P′ such that $NP_1 = PN$, then P_1 is the image of P.



The image of arc PQR will be the arc $P_1Q_1R_1$.

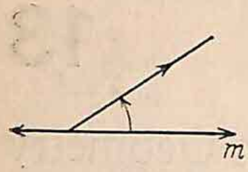
To get the image practically, we can fold the paper about the mirror line and

(*i*) stick a number of pins along PQR, unfold the paper and join the pin points.

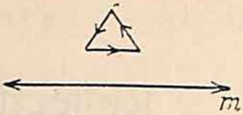
or (*ii*) use a carbon paper.

or (*iii*) use a tracing paper.

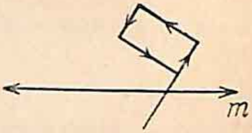
The students can find the images of the following figures in the given mirror lines.



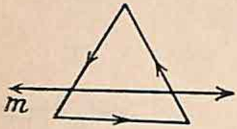
(*i*)



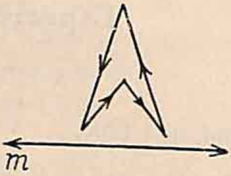
(*ii*)



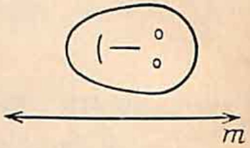
(*iii*)



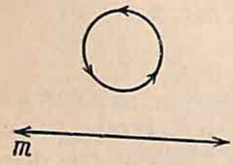
(*iv*)



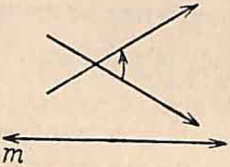
(*v*)



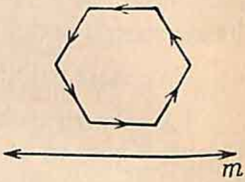
(*vi*)



(*vii*)



(*viii*)



(*ix*)

From these, the students can deduce the following conclusions:

(*i*) By reflection, a straight line remains a straight line, a triangle remains a triangle, a quadrilateral remains a quadrilateral, a circle remains a circle and so on.

(*ii*) If the original figure (triangle, square, circle etc.) is described in the counter clockwise sense, then the image figure is described in the clockwise sense and *vice versa*.

(*iii*) Length of a line in the original figure is the same as in the image figure.

(*iv*) The angle between two lines on the original figure is the same as the angle between the corresponding lines in the image figure, though the sense is different.

(*v*) Points on the mirror line do not change by reflection.

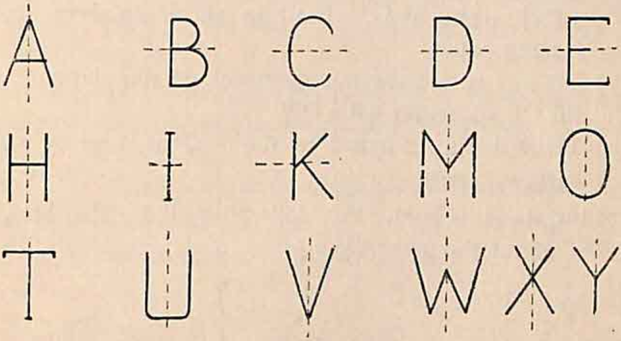
(*vi*) Lines which are parallel to the mirror remains parallel to the mirror after reflection.

(*vii*) The original line and the reflected line make equal angles with the mirror line, though the senses of the angles are different.

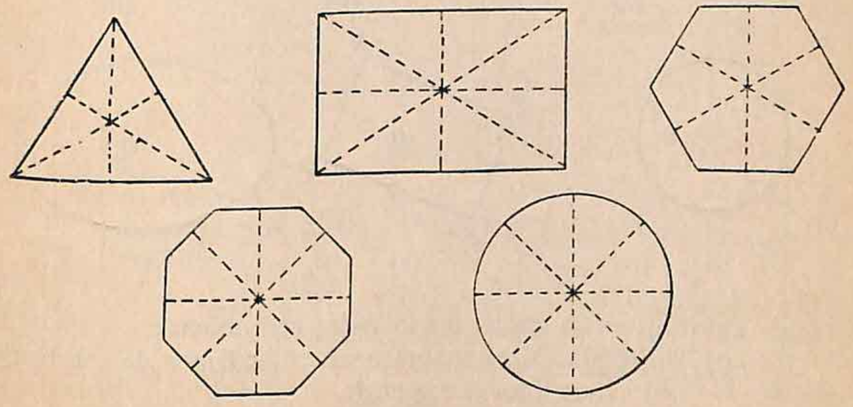
(*viii*) If we reflect back in the mirror line, we get the original figure.

Experiment 419. Figures Which do not Change by Reflection.

Some figures do not change after reflection in a line *e.g.* consider reflection in dotted lines.

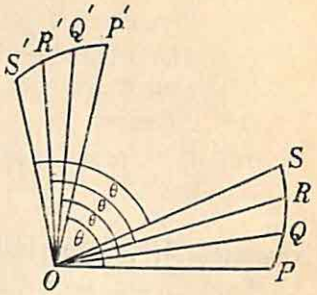


Such lines are called axes of symmetry. A figure may have more than one axis of symmetry. How many axes of symmetry does a circle have ? Consider also axes of symmetry in the following figures.



Experiment 420. Rotation About a Point.

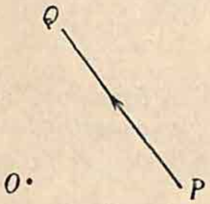
To rotate a figure PQRS about a fixed point O in a plane through an angle θ, we rotate each of the lines OP, OQ, OR, OS through the same angle θ to get the points P′, Q′, R′, S′ of the image figure.



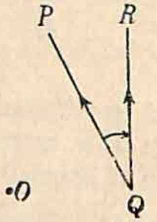
To find the image figure, the student can proceed as follows :

- (*i*) Let him take a point P of the figure. Join OP. Draw OP′ making an angle θ with OP.
- (*ii*) Let him put a tracing paper on the plane of the paper and attach a pin at O. Let him trace the given figure on the tracing paper.
- (*iii*) Let him rotate the tracing paper on the plane of the paper till OP coincides with OP′.
- (*iv*) Trace the given figure on the original sheet by using pins or carbon paper.

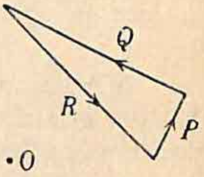
Let the students rotate the following and other similar figures through 60° about the point O.



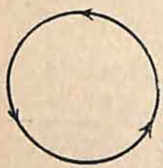
(*i*)



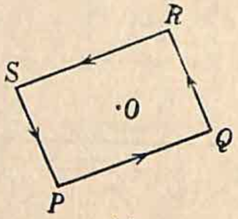
(*ii*)



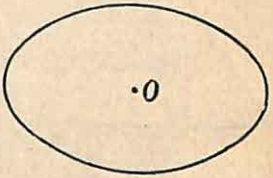
(*iii*)



(*iv*)



(*v*)



(*vi*)

Let the students deduce the following conclusions :

- (*i*) The lengths, sizes, shapes, areas of the figures do not change by rotation through any angle.

(*ii*) The angles between lines do not change.

(*iii*) If we rotate *back* through an angle θ, we get the original figure.

(*iv*) If we first rotate through an angle θ_1, then rotate through an angle θ_2 about the same point or if we first rotate through an angle θ_2 and then through an angle θ_1 about the same point, the final figure is the same.

(*v*) The orientation of figures and the angles of intersection between figures do not change by rotation.

(*vi*) The point O is the only point which does not change by rotation about the point O.

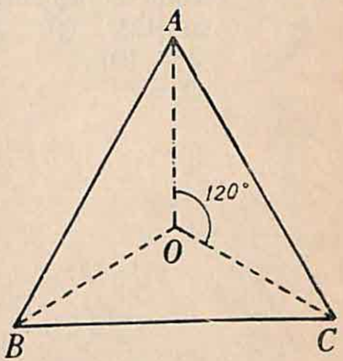
(*vii*) OP′=OP, OQ′=OQ etc. so that the distance of any image point from O is the same as the distance of the original point from O.

(*viii*) A ray and its image after half-turn (rotation through 180°) are parallel and point in opposite directions.

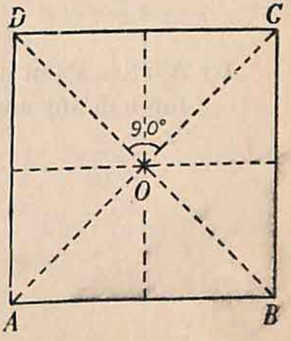
Experiment 421. Figures which do not Change by Rotation.

Some figures do not change by rotation about suitable points through suitable angles *e.g.*,

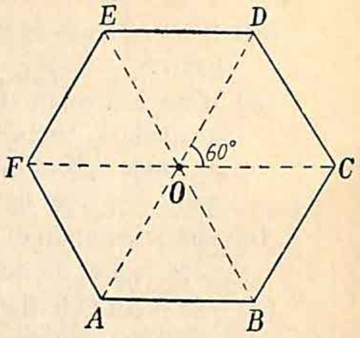
(*i*) An equilateral triangle about its centre through angles of 120°, 240°, 360°.



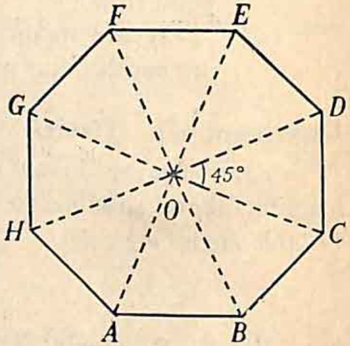
(*ii*) A square about its centre through angles of 90°, 180°, 270°, 360°.



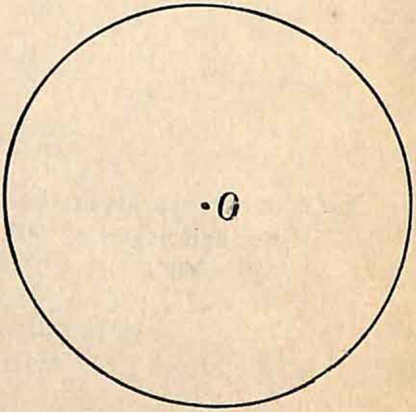
(*iii*) A regular hexagon about its centre through angles of 60°. 120°, 180°, 240°, 300°, 360°.



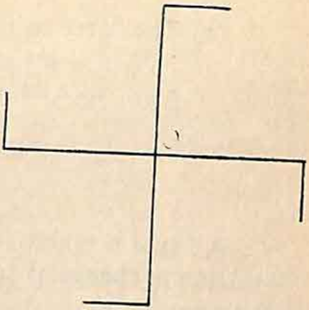
(*iv*) A regular octagon about its centre through angles of 45°, 90°, 135°, 180°, 225°, 270°, 315°, 360°.



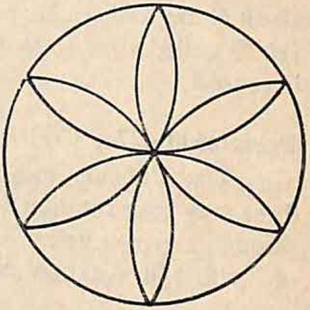
(*v*) A circle about its centre through any angle.



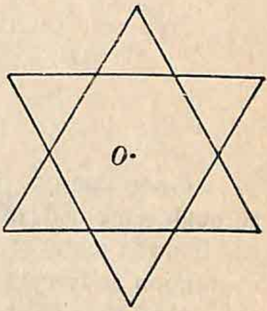
(*vi*) A swastika about its centre through 90°, 180°, 270°, 360°.



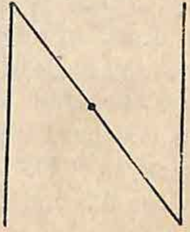
(*vii*) The figure on the right about its centre through angles of 60°, 120°, 180°, 240°, 300°, 360°.



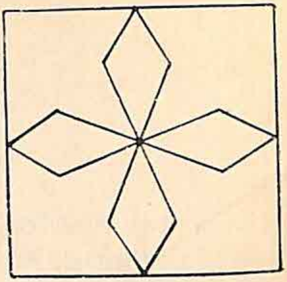
(*viii*) The figure on the right about its centre through angles of 60°, 120°, 180°, 240°, 300°, 360°.



(*ix*) The letter N about its centre through 180°.



(*x*) The figure on the right about its centre through 90°, 180°, 270°, 360°.

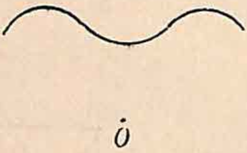


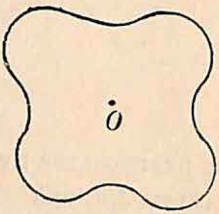
A figure is said to have a *rotational symmetry* about a point O, if it does not change by rotation through some angle or angles about that point.

Those figures which remain unchanged on rotation through 180° about O are said to have *point symmetry* about O. Rotation about 180° is called a *half-turn* and with this transformation, all rays change in sense.

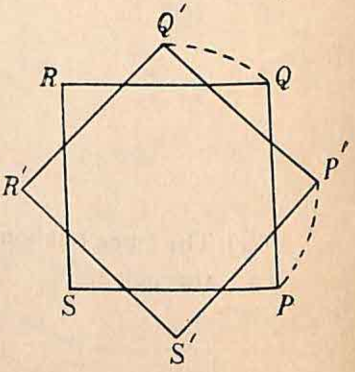
Experiment 422. Centre of Rotation

The students may be asked to complete given figures so that these may have rotational symmetry about a given point for rotation through a given angle *e.g.* the second figure below gives the completion of the first for rotation about O through 90°.





Given that one figure is obtained by rotation about some point through some angle of another figure, we can find the centre of rotation and angle of rotation as follows :



Find the right bisectors of PP′ and QQ′. Their point of intersection gives the centre of rotation and POP′ or QOQ′ gives the angle of rotation. We may get other possible centres and angles of rotation by finding intersection of right bisectors of PQ′ and QR′ or PR′ and QS′ or PS′ and QP′.

The same problem may be tried for congruent pairs of equilateral triangles, rectangles, hexagons etc.

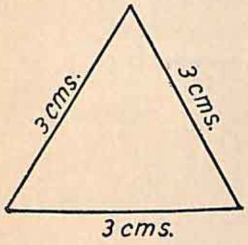
The concept of point symmetry or central symmetry may be emphasized *e.g.* a circle has central symmetry about its centre and about no other point. What are the points of central symmetry, if any, of a square, a rectangle, a parallelogram, a rhombus, a regular hexagon, a regular octagon, an ellipse etc. ?

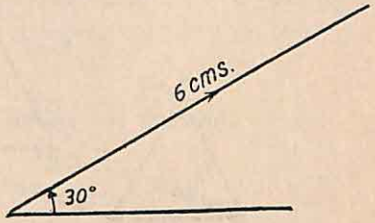
Experiment 423. Translation

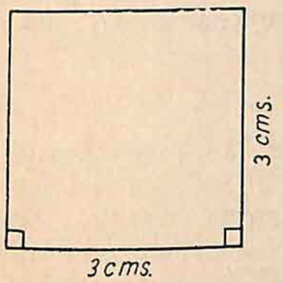
Reflection is a rigid body transformation which is known if the axis of reflection is specified. Rotation is a rigid body transformation which is known if the centre of rotation and angle of rotation are known. In both these transformations, the distances between points and angles between lines do not change by the transformations and mappings, though in the case of reflection, the sense of rotation of the angles changes.

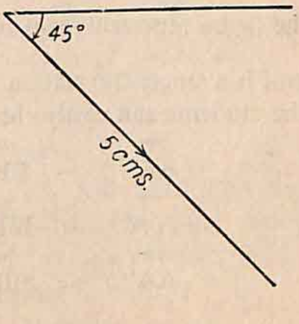
A third rigid body transformation which does not change distances or angles is *translation* which is obtained when a figure is moved in a given direction through a given distance.

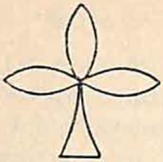
Let the students obtain the maps of the following figures for the specified translation.

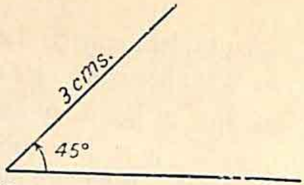




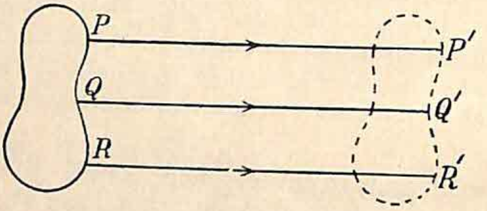




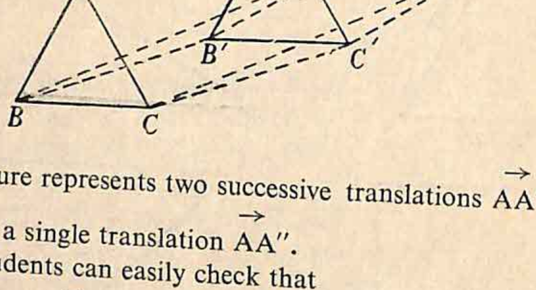




To find the image of a given figure by a given translation, we take three points P, Q, R not in the same straight line in the given figure, draw lines of given length from these points in the given direction to get P′, Q′, R′, trace the given figure on another piece of paper and place the tracing so that P, Q, R coincide with P′, Q′, R′ and then trace the image by using carbon paper or pins.



Experiment 424. Combination of Two Translations



The figure represents two successive translations $\overrightarrow{AA'}$ and $\overrightarrow{A'A''}$. The result is a single translation $\overrightarrow{AA''}$.

The students can easily check that

$$\overrightarrow{AA'} = \overrightarrow{BB'} = \overrightarrow{CC'}$$

$$\overrightarrow{A'A''} = \overrightarrow{B'B''} = \overrightarrow{C'C''}$$

$$\overrightarrow{AA''} = \overrightarrow{BB''} = \overrightarrow{CC''}$$

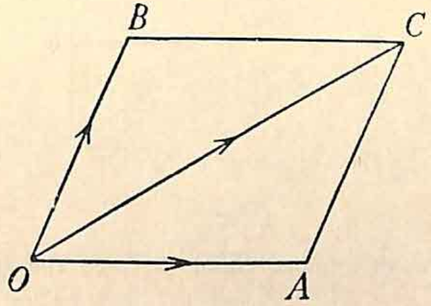
i.e. they check that the directions of AA′, BB′, CC′ are parallel and the magnitudes of AA′, BB′, CC′ are equal and that similar results hold for other segments.

The students can verify by drawing a number of figures that

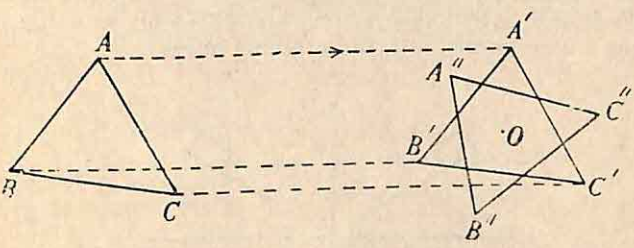
(*i*) the resultant of two translations is a translation.

(*ii*) the resultant translation is obtained by the "triangle law" according to which if two translations are represented by two sides $\overrightarrow{AA'}$ and $\overrightarrow{A'A''}$ of a triangle, the resultant translation is obtained by the translation $\overrightarrow{AA''}$.

(*iii*) the resultant translation can also be obtained by the "parallelogram law" according to which if the translations are represented by the two sides OA and OB of a parallelogram, the resultant translation is represented by the diagonal OC of the parallelogram.



Experiment 425. Product of Translation and Rotation

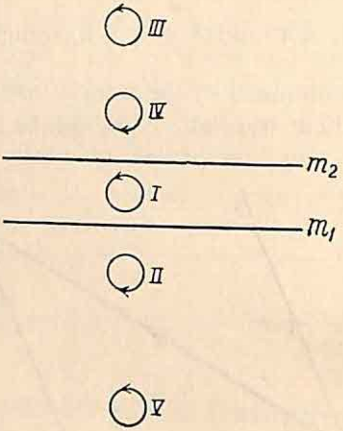


Take a triangle ABC, translate it to triangle A′B′C′ and rotate it to position A″ B″ C″. Join AA″, BB″, CC″ and find their right bisectors. All of them will meet in a common point, showing that the resultant of a translation and a rotation is a rotation about some suitable point through some suitable angle.

Let the students verify the result for a number of cases and let them also check whether a translation followed by a rotation gives the same result as a rotation followed by a translation.

Experiment 426. Product of Two Reflections

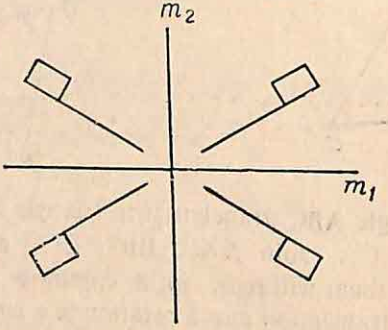
By reflection in m_1, figure I goes into figure II and by reflection in parallel mirror m_2, figure II goes into figure III. The net result is a translation through distance equal to twice the distance between the mirrors.



On the other hand, if the figure is first reflected in m_2 and then in m_1 figure V is obtained, so that a translation through the same distance but in opposite direction is obtained.

Thus the product of reflection in two parallel mirrors is a translation and the operation is not commutative.

The results of combination of reflections can be easily obtained by folding along parallel lines on a tracing paper

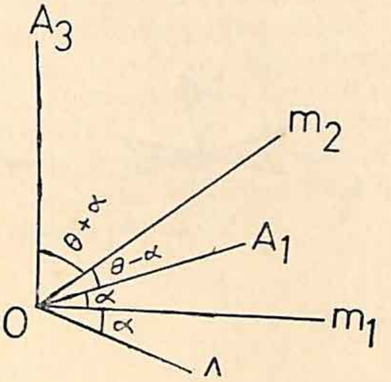


If the two mirrors are at right angle, the product is the half-turn about the point of intersection. The reflection may be taken in any order.

If the angle is not a right angle, the resultant would still be a rotation since only one point O remains unchanged.

Let the image of point A_1 in m_1 be A_2 and let the image of A_2 in m_2 be A_3 then $\angle A_3 O A_1 = 2\theta$

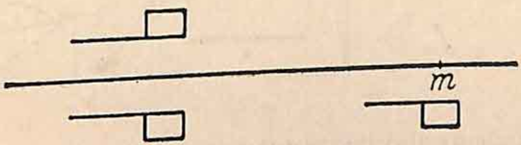
$= 2 \times$ angle between mirrors.



Thus the product of two reflections in two intersecting mirrors is a rotation through an angle equal to twice the angle between the mirrors. Also the product of reflection in m_2 and then in m_1 is different from the product of reflection in m_1 and then in m_2.

Experiment 427. Product of a Reflection and Translation

The combination of a reflection in some line and a translation in the same direction is equivalent to a reflection or a combination of a reflection in a suitable line and a translation parallel to the same mirror.



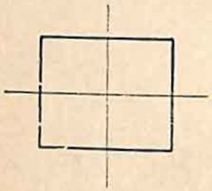
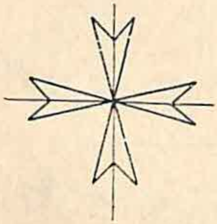
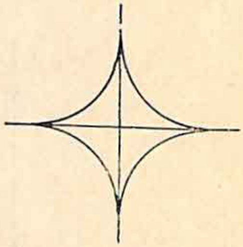
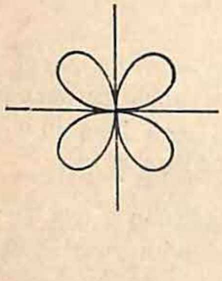
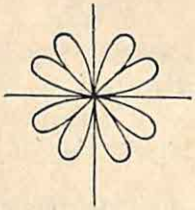
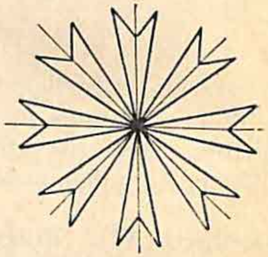
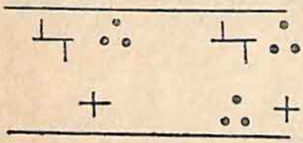
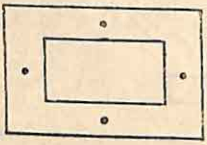
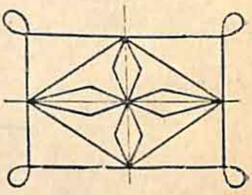
This combination is called a *glide reflection.*

Experiment 428. Symmetry

One of the most important aspects of the study of mathematics is the study of patterns in numbers and space. The study of patterns in space is greatly facilitated by the study of symmetry of figures. The symmetry is studied in terms of those transformations which leave the figure unchanged. The transformations we examine are :

(*i*) reflections
(*ii*) translations
(*iii*) rotations including half turn
(*iv*) glide reflections.

We give below some figures which show some symmetries. Let the students find these symmetries.

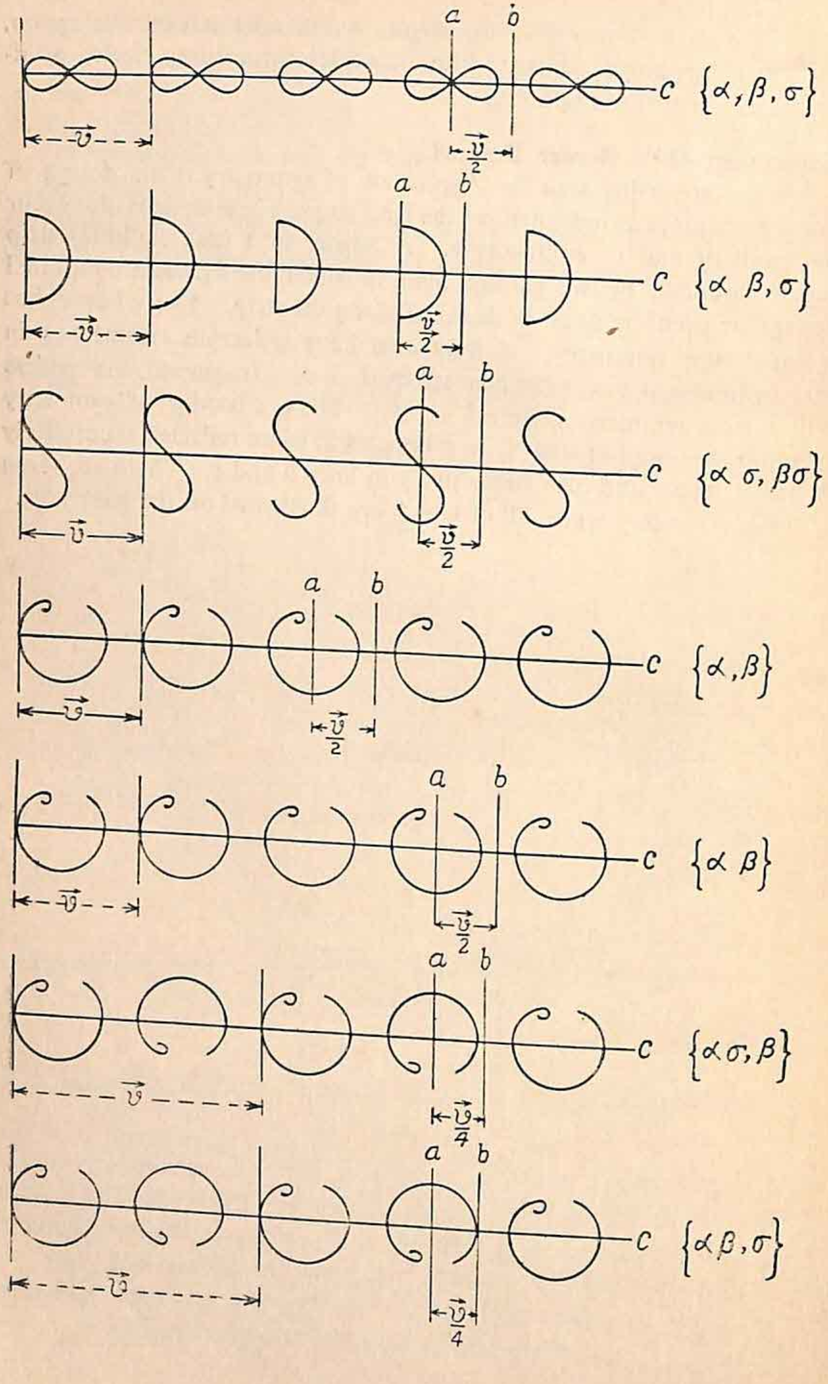










Let the students also find symmetries in all the letters of the alphabet and all the playing cards. They can also look for symmetries in ornaments, paintings, architecture, flowers, animals and crystals. They can also find points of symmetry, lines of symmetry, planes of symmetry, rotational symmetries in the regular solids *viz*. cube, tetrahedron, octahedron, icosahedron and dodacahedron.

The student may also investigate symmetries in a parallelogram, rhombus, rectangle, square, kite, isosceles trapezium, circle, semicircle, equilateral triangle etc.

Experiment 429. Border Designs

An interesting area for observation of symmetry is in design of border of sarees as indicated on the next page. These designs also occur in jewellery and in architecture. A border is a long (infinite) strip region bounded by two parallel lines in which the figures are repeated at regular equal intervals of distance along the strip. Every border has a translation symmetry. It may also have reflection symmetries in certain lines $a, b, c, \ldots$ to be denoted by $\alpha, \beta, \sigma, \ldots$ In general, the borders will possess symmetries like $\alpha\beta$, $\alpha\sigma$ etc. Thus a border of symmetry type ($\alpha\sigma$, $\beta\sigma$) implies that it is symmetrical when reflected successively in lines a and c and also successively in lines b and c. There are seven of these symmetry types, all of which are illustrated on the next page.


a
b
c
$\{\alpha, \beta, \sigma\}$
$\vec{v}$
$\frac{\vec{v}}{2}$
a
b
c
$\{\alpha\ \beta, \sigma\}$
$\vec{v}$
$\frac{\vec{v}}{2}$
a
b
c
$\{\alpha\ \sigma, \beta\sigma\}$
$\vec{v}$
$\frac{\vec{v}}{2}$
a
b
c
$\{\alpha, \beta\}$
$\vec{v}$
$\frac{\vec{v}}{2}$
a
b
c
$\{\alpha\ \beta\}$
$\vec{v}$
$\frac{\vec{v}}{2}$
a
b
c
$\{\alpha\sigma, \beta\}$
$\vec{v}$
$\frac{\vec{v}}{4}$
a
b
c
$\{\alpha\beta, \sigma\}$
$\vec{v}$
$\frac{\vec{v}}{4}$

14

Shearing, Area and Volume

Experiments : 430 to 439

Experiment 430. Shearing as a Transformation.

So far we have considered the following simple transformations or mappings :

(*i*) reflection in a line (*ii*) rotation about a point
(*iii*) translation (*iv*) glide reflection.

Each of these is a one-to-one correspondence which preserves shape and size, though orientation may change. Each transformation maps straight lines into straight lines and does not change angles between straight lines.

There are other transformations which may not have these properties *e.g.*

(*i*) photograph of a three-dimensional object on a plane film by a camera. This is a many-one mapping since all points on the line of sight of camera lens are mapped into a single point on the film. Such straight lines are mapped into points. Other lines may be mapped into lines, but angles and sizes are not necessarily preserved.

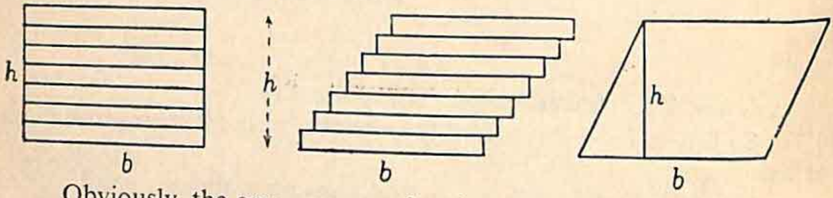
(*ii*) image by a pinhole camera. This preserves the shape, but changes the size and inverts the figure.

(*iii*) figures drawn on a rubber sheet are mapped onto other figures by stretching the rubber sheet without tearing. Such mappings are one-one and preserve continuity but do not preserve size or shape. Such transformations are called topological transformations.

(*iv*) image by shearing where we keep a line or a plane in the body fixed (*i.e.* its image is itself) and transformation is such that the distance of the image point P′ from the line (or plane) is the same as the distance of the original point P for the line (or the plane) so that lines (or planes) parallel to the line (or the plane) remain parallel to the line (or the plane) after the transformation. Shearing changes shape but not area or volume.

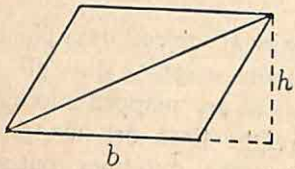
Experiment 431. Use of Shearing to Get Formulae for Area

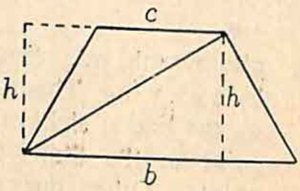
The following figures, give a side view of a pile of thick cards before and after shearing :



Obviously the area we see of each card remains the same and therefore the total area of the cards in the new position is the same as before. If we take the cards very thin, we shall get a parallelogram as shown in the figure. The area of the parallelogram is therefore the same as the area of the rectangle so that

area of the parallelogram = base × height





The diagonal of a parallelogram divides it into two equal triangles. The areas of the two triangles are equal and each one is equal to half the area of the parallelogram. Thus

area of the triangle $= \frac{1}{2}$ base × height $= \frac{1}{2}\, b.h.$

A trapezium is a quadrilateral with two parallel sides. Its diagonal divides it into two triangles whose areas are $\frac{1}{2}\, bh$ and $\frac{1}{2}\, ch$ so that the area of the trapezium $= \left[\frac{1}{2}(b+c)\right] h = \frac{1}{2}$ (sum of parallel sides) × distance between the parallel sides.

Experiment 432. Use of Shearing to Find Formulae for Volume

Consider now the volume of a pile of cards. For the cuboid the volume is area of the base × height and as such the volume of the parallelopiped which is obtained by shearing the cuboid so that its base remains fixed and planes originally parallel to the base rmain parallel to the base, is also given by the same formula.

Experiment 433. Finding Formulae for Areas by Using Graph Paper

Let the students draw rectangles of different lengths and breadths on a graph paper and by counting the number of squares in each rectangle, let them deduce that

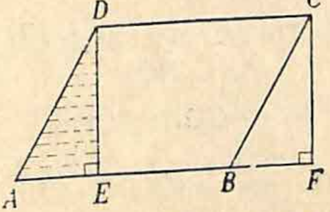
(*i*) for a fixed breadth, the area is proportional to the length
(*ii*) for a fixed length, the area is proportional to the breadth
(*iii*) the area of a rectangle = its length × its breadth.

Similarly let the students draw triangles of different lengths of bases and of different heights, count the number of full squares and of squares which are more than half inside the triangles and deduce that

(*i*) for a fixed length of base, the area is proportional to the height ;
(*ii*) for a fixed height, the area is proportional to the length of the base ;
(*iii*) the area of a triangle $= \frac{1}{2}$ base × height.

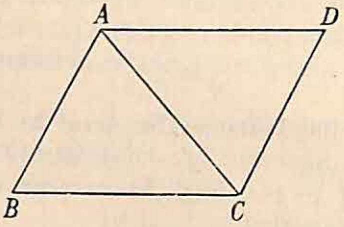
Experiment 434. Finding Formulae for Areas by Cutting and Pasting

To find the area of a parallelogram ABCD, let the students cut off the triangle AED and paste it on the position BCF to get the rectangle EFCD so that



Area of parallelogram ABCD = area of rectangle EFCD
= EF × CF = base × height

To find the area of the triangle ABC, make another equal triangle and paste it in the position ACD to get the parallelogram

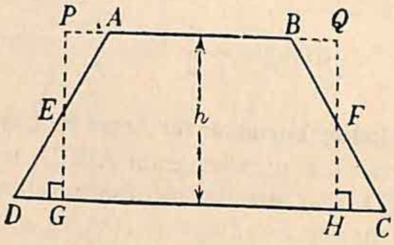


ABCD so that area of the triangle ABC

$$= \frac{1}{2} \text{ area of parallelogram ABCD}$$

$$= \frac{1}{2} \text{ base} \times \text{height.}$$

To find the area of a trapezium ABCD, let E and F be the middle points of AD and BC. Cut off right-angled triangles EGD and FHC and put these in the positions AEP and BFQ so that



area of trapezium ABCD = area of rectangle GHQP

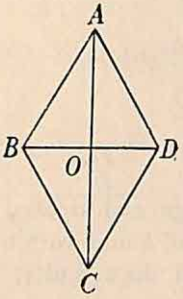
$$= \text{GH} \times \text{HQ}$$

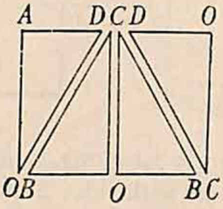
$$= \frac{1}{2} (\text{GH} + \text{PQ}) \times h$$

$$= \frac{1}{2} (\text{DC} - \text{DG} - \text{CH} + \text{AB} + \text{AP} + \text{BQ}) \times h$$

$$= \frac{1}{2} (\text{AB} + \text{CD}) \times h$$

Similarly to find the area of a rhombus ABCD, we cut off the triangles AOB, AOD, BOC, COD and place these as shown to get a rectangle whose sides are 2OD and OA so that



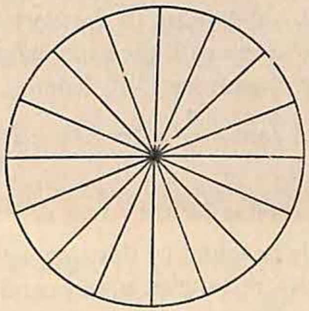


area of rhombus$=2\text{OD}\times\text{AO}=\frac{1}{2}$ product of diagonals

Experiment 435. Formula for Area of a Circle.

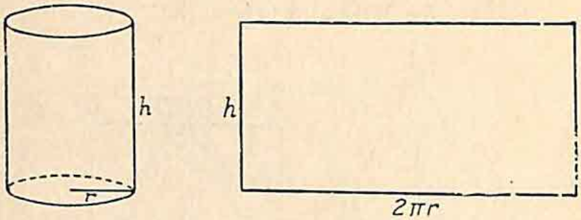
We assume that the ratio of the circumference of a circle to its diameter is the same for all circles. This number is a fixed number π whose approximate value is $\frac{22}{7}$.

To find the area of a circle, we divide the circumference into a large number of parts and join the points to the centre to get a large



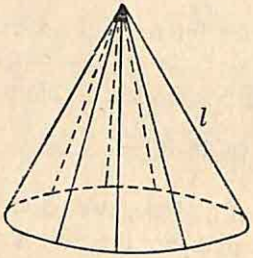
number of triangles. The area of each triangle is $\frac{1}{2}$ base × height. The height in every case is the radius of the circle and the sum of lengths of bases is $2\pi r$ so that the total area of the circle is $\frac{1}{2}\times 2\pi r\times r=\pi r^2$.

Experiment 436. Formula for Areas of Curved Surfaces of Cylinders and Cones.



If a cylinder is cut along an edge and opened out, its surface becomes a rectangle with sides $2\pi r$ and h and with total area $2\pi r \times h$ which gives the curved surface area of the cylinder.

For the curved surface area of a right circular cone, we again divide the circumference of the base into a large number of parts



and join the points of subdivision to the vertex of the cone to get a number of triangular areas with the same height l which is the slant height of the cone and with sum of lengths of bases $2\pi r$ so that curved surface area of cone$=\frac{1}{2}\times 2\pi r \times l = \pi r l$.

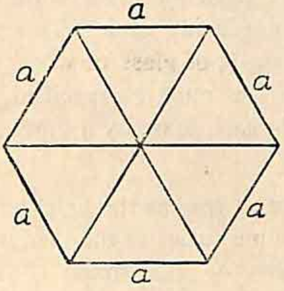
Experiment 437. Formulae for the Areas of Surfaces of Polyhedra.

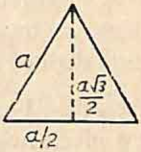
These can easily be found by drawing nets of the polyhedra. The nets consist of triangles, rectangles, squares and polygons. The formulae for the areas of triangles, rectangles and squares are known. A polygon can be divided into triangles and the sum of the areas of the triangles gives the area of the polygon. Thus surface area of the cube of edge $a = 6a^2$.

The students can verify that the surface area of a cube is $6a^2$ by pasting 1″ square pieces on its sides.

The surface area of a regular tetrahedron of edge a is equal to four times the area of an equilateral triangle of side a so that surface area of tetrahedron$=4\times\frac{1}{2}a\times\frac{a\sqrt{3}}{2}=\sqrt{3}a^2$. Similarly the surface area of a hexagonal prism of edge a and height $h=$area of 2 hexagons of edge $a+$ area of 6 rectangles of area $ab=2\times6\times\frac{1}{2}\,a\times a\,\frac{\sqrt{3}}{2}+6ab$

$$=3\sqrt{3}a^2+6ab$$





The students may find the surface areas of all the regular polyhedra they have constructed.

Experiment 438. Volumes of Polyhedra.

(*a*) **By Centimetre Cubes.**

Let the students construct cardboard models of open cubes of edges 3 cm, 4 cm, 5 cm and let them be given centimetre cubes and let them find that 27, 64 and 125 cubes respectively are required to fill these cubes. Let them notice the pattern and deduce the formulae for the volume of a cube.

Similarly let them construct cardboard models of cuboids (or they can be given open wooden cuboids) of different dimensions and they can find that the volume of a cuboid or a rectangular parallelopiped is given by length × breadth × height or by the area of the base × height.

(*b*) **By Measuring the Volume of Water Required to Completely Fill the Polyhedra.**

Let the students be given open hollow prisms of different bases and different heights and let them fill these with water and measure the volume of the water with the help of a graduated cylinder. Let them deduce that

- (*i*) For a given base, volume varies as the height of the prism
- (*ii*) For a given height, volume varies as the area of the base
- (*iii*) Volume is independent of the shape of the base, but depends on its area
- (*iv*) Volume of a prism = area of the base × height.

Similarly let the students be given pyramids with different bases and different heights, hollow from inside and with a small hole at the vertex. The models may be made of glass or wood or cardboard. In each case, the internal dimensions must be specified. The volume of the water may be measured in each case by a graduated cylinder. Let the students deduce that

- (*i*) for a given base, volume varies as the height of the pyramid
- (*ii*) for a given height, volume varies as the area of the base
- (*iii*) volume is independent of the shape of the base, but depends only on its area
- (*iv*) volume is independent of the position of the vertex on a plane parallel to the base at a fixed height
- (*v*) volume $= \frac{1}{3}$ area of the base × height.

(*c*) By Other Methods

Consider the six pyramids fitting inside a cube of edge a. The volume of the cube is a^3. Therefore the volume of the pyramid is $\frac{a^3}{6}$. The area of the base is a^2 and its height is $\frac{a}{2}$. This verifies that the volume of this pyramid is $\frac{1}{3}$ area of base × height.

The student can make plastericine models of various polyhedra with different dimensions. They can then make the same materials into cuboids with bases of edges 1 cm or 2 cm or 3 cm and measure the heights and thus determine the volumes. They can then deduce all the formulae earlier deduced by means of a graduated cylinder.

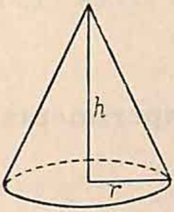
The children can also be given solid models made of wood or glass of various sizes. They can immerse these in a graduated cylinder half filled with water, note the change in volume and deduce all the formulae for volumes.

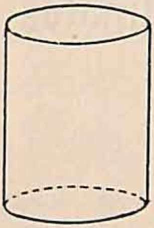
Experiment 439. Volumes of Cylinders and Cones.

A cylinder can be regarded as a special case of a prism when the base is circular and as such its volume=area of base × height $=\pi r^2 h$.

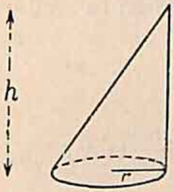
A cone can be regarded as a special case of a pyramid when the base is circular and as such its volume $=\frac{1}{3}$ area of base × height

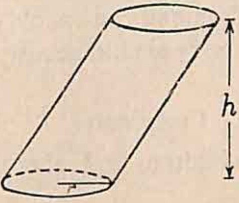
$$=\frac{1}{3}\pi r^2 \times h$$





The volume does not change by shearing and as such the volumes of the solids shown are also given by similar formulae.





15

Co-ordinates in Three Dimensions

Experiments : 440 to 453

The following experiments show how coordinate geometry of three dimensions can be introduced at quite an early stage in school and that such an introduction can considerably strengthen the spatial intuition of children and can enable children to study two and three dimensional geometry simultaneously.

Experiment 440. Coordinates.

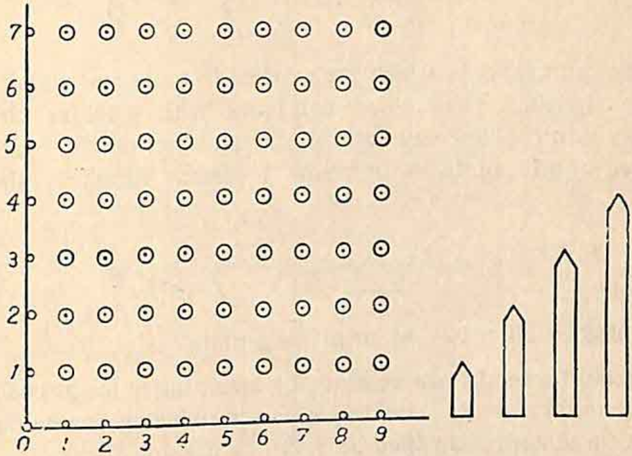
Let the children find the truth set of the open sentence

$$\square + \triangle + \text{⬡} = 6$$

when the universal set or the substitution set is the set of whole numbers. They get ordered triplets like the following :

(0, 1, 5), (1, 5, 0), (1, 0, 5), (0, 5, 1), (5, 1, 0), (5, 0, 1),
(0, 2, 4), (2, 4, 0), (2, 0, 4), (0, 4, 2), (4, 2, 0), (4, 0, 2),
(1, 1, 4), (1, 4, 1), (4, 1, 1), (1, 2, 3), (1, 3, 2), (2, 3, 1),
(2, 1, 3), (3, 1, 2), (3, 2, 1), (2, 2, 2), ...

For representing ordered pairs, they had used crossed number lines and the truth sets were represented on a graph paper. How can they represent ordered triplets ? The following board can be used :



There are small holes at each of the lattice points and children are given small sticks of lengths 1, 2, 3, 4, 5, 6, ... units which can just fit in the holes. To represent the ordered triplet (1, 2, 3), they can find the lattice point (1, 2) and then place the stick of lengths 3 vertically in it. To represent (1, 5, 0) no stick is placed in the lattice point (1, 5).

Instead of a board with holes and wooden sticks, a magnetic board with no holes and iron rods can also be used.

Experiment 441. Equation of a Plane.

Let the children represent all the ordered triplets which are members of truth set of □ + △ + ⬡ = 6. Let them now take a plane cardboard and place it on the sticks or rods. It fits them exactly. The truth set of this sentence determines a plane.

Similarly let the children verify that the truth sets of open sentences,

□ + △ − ⬡ = 4, □ − △ − ⬡ = 4,2 □ + △ + ⬡ = 6

all determine planes. Which types of open sentences determine planes ?

Now consider the open sentence

□ + △ + ⬡ ⩽ 6

They can use sticks which are marked in units and which are all of the same height. The children can mark with a colour chalk the points they want. They will see that the truth set of this sentence is represented by all points on or below a plane. Similarly the truth set of

□ + △ + ⬡ ⩾ 6

is represented by all points on or above a plane.

A beautiful model can be made by attaching bulbs at each marked point on each rod and providing separate switches for them. The truth set of a sentence can then be 'seen' by lighting the bulbs corresponding to it.

Experiment 442. Equations of a Straight Line.

Consider the open sentences.

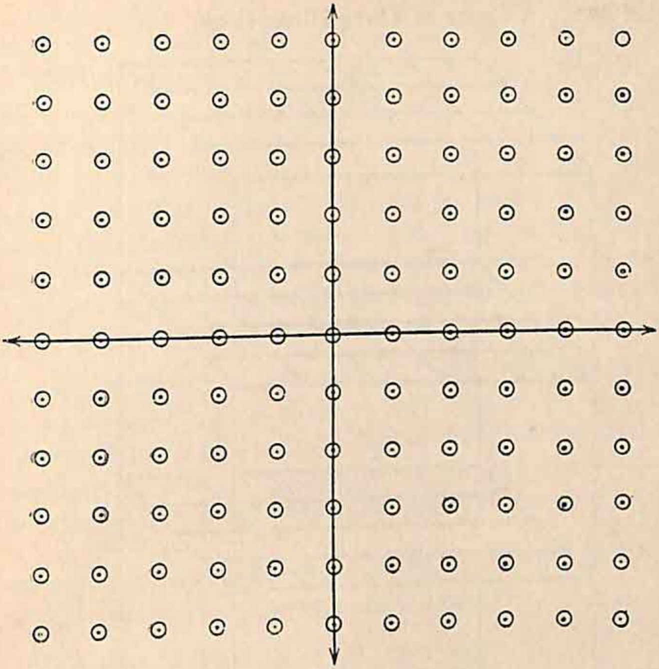
□ + △ + ⬡ = 6 and □ − △ + ⬡ = 4

Let the children mark the truth set of the first sentence by red chalk marks and mark the truth set of the second sentence by blue chalk marks. They get some points with both red and blue chalk marks. Let them join these points by a thread. The thread will be found to be straight. Thus the truth set of the two sentences together represents points lying on a straight line.

The children can take many more such pairs of equations of the first degree *i.e.* in which there are no squares or products of frames and verify that they all determine straight lines.

Experiment 443. Negative Co-ordinates.

The truth sets of the above sentences can also be considered when the universal set or the substitution set is the set of integers. In this case we can use wooden or glass board (as shown below) with rods fixed rigidly at all lattice points so that the rods project both above and below and carry bulbs or marks at unit intervals.



The children will find that even in this case, an equation of the first degree represents a plane and two equations of the first degree together represent a straight line.

Experiment 444. Locating the Position of Objects.

The children may use three co-ordinates to locate the position of objects in space *e.g.* for the following :

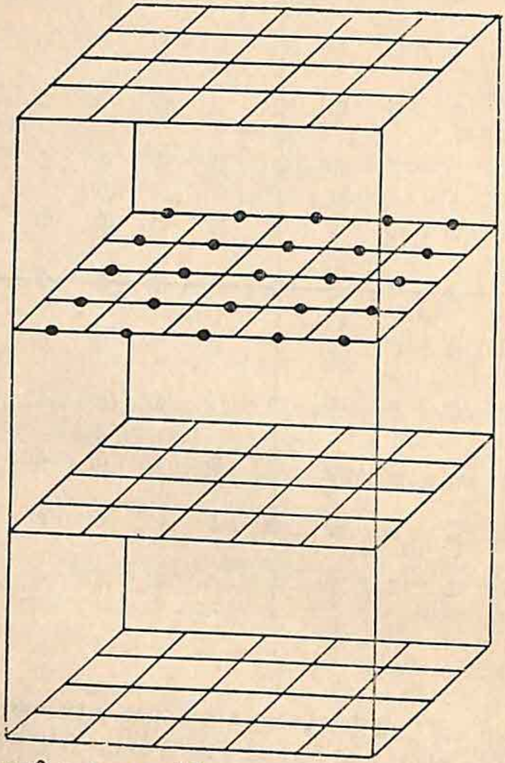
(*i*) locating the position of a light bulb in the class room

(*ii*) locating the position of a ball when balls are placed in rectangular arrays in a number of horizontal trays placed vertically over one another

(*iii*) the position of an aeroplane in air

(*iv*) the position of a submarine in the sea.

Experiment 445. A Game in Three Dimensions.



There are four trays with sixteen holes for keeping balls. These are arranged as shown above. Two children play with red and blue balls. The first child to complete a straight line in space wins.

Experiment 446. Intersection Set of Truth Sets.

The children can verify that the intersection set of truth sets of $\square-\triangle+\hexagon=4$ and $\square-\triangle-\hexagon=4$ represents a straight line, the intersection set of $\square+\triangle+\hexagon=4$ and $\square+\triangle+\hexagon=5$ is the null set, the intersection set of $\square+\triangle+\hexagon=6$, $\square-\triangle+\hexagon=2$, is the singleton {3, 2, 1}, the intersection set of $\square+\triangle+\hexagon=6$, $2\,\square+\triangle+2\hexagon=8$ and $3\square+2\,\triangle+3\hexagon=14$ is a straight line and so on.

Experiment 447. Coordinates of Points on a Straight Line

A tight string, with knots at regular intervals may be streched from one corner of the roof of the classroom to the opposite corner of the floor of the classroom and children may measure the coordinates of different knots on the string. For this purpose they may use a plumb line with a light weight to find the projection of the knots on the floor. The coordinates of the projection on the floor together with the height of the knot above the floor will determine the coordinates of the knots. These coordinates need not be whole numbers, but since these will be obtained by measurement, they will be fractions.

Let the children find a number of linear relations between the coordinates of points on the string. They should be able to find a number of such relations, but only two of these will be independent in the sense that the others can be deduced from these by multiplying these by suitable numbers and then adding or subtracting.

Experiment 448. Open Sentences in Four or More Variables.

Let the children consider the open sentence

□ + △ + ⬡ + ▱ = 6

The truth set in the domain of whole numbers consists of ordered quadruples like

(0, 1, 2, 3), (1, 0, 2, 3), (1, 2, 0, 3), (1, 2, 3, 0),
(0, 2, 3, 1), (2, 0, 3, 1), (2, 3, 0, 1), (2, 3, 1, 0),
(0, 2, 2, 2), (2, 0, 2, 2), (2, 2, 0, 2), (2, 2, 2, 0),
(1, 1, 2, 2), (1, 2, 1, 2), (2, 1, 2, 1), (2, 2, 1, 1), ...

The children can find the truth set, but they cannot represent geometrically, because they cannot find a 'fourth' dimension. They can however 'imagine' a four-dimensional space and say that the above equations represent a 'plane' (we call it a hyperplane) in this space. Similarly

□ + △ + ⬡ + ▱ + 3 ◇ = 10

represents a 'hyperplane' in 'five dimensional space'.

Experiment 449. A Paraboloid of Revolution.

Consider the open sentence

$$x^2+y^2=z$$

and let the universal set be I. The children can get a large number of ordered triplets as truth sets of the sentence *e.g.* following are members of truth set

(0, 0, 0), (0, 1, 1), (1, 1, 2), (1, 2, 5) (−1, −2, 5),
(1, −2, 5), (−1, 2, 5), (−1, −1, 2) (1, −1, 2) (2, 2, 8),
(2, −2, 8), (−2, 2, 8), (1, 3, 10), (1, −3, 10) etc.

Let the children draw these points and note these do not lie in a plane but on a 'curved' surface which is called a paraboloid of revolution.

They can get other paraboloids of revolution by considering the open sentences

$$x^2+y^2=kz$$

where k is any number.

Experiment 450. A Right Circular Cone.

Consider the open sentence

$$x^2+y^2=z^2$$

and let the universal set be I. The children can find the following and other members of the truth set

(1, 0, 1), (0, 1, 1), (3, 4, 5), (−3, 4, 5), (3, −4, 5), (−3, −4, 5),
(6, 8, 10) (−6, 8, 10), (6, −8, 10) (−6, −8, 10), ...

The surface obtained in a right circular cone. The children can make a paper cone and try to fit it and then measure the semi-vertical angle of the cone.

Experiment 451. Finding the Equations of a Surface.

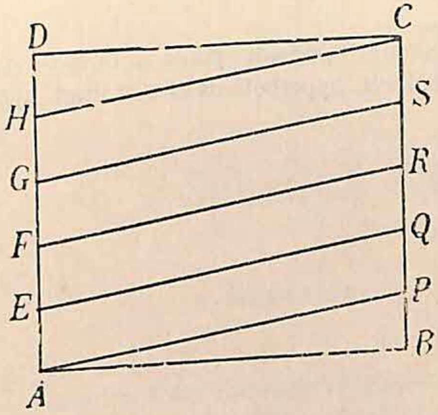
Give a wooden right circular cone of semi-vertical angle 45° and let the children place it with its axis vertical and vertex at the origin on the wooden board described above. The children can measure the coordinates of many points on the surface and then try to guess a relation between them. They would find $x^2+y^2=z^2$. They can find other right circular cones with different semi-vertical angles. with the z-axis as axis find co-ordinates of many points on each and find that $(x^2+y^2)/z^2$ is constant in each case.

Similarly let them place an inverted hemispherical cap with its vertex vertically above the origin, let them measure the coordinates of many points and check that $x^2+y^2+z^2=$const.

The same experiment can be repeated with paraboloids of revolution, hyperboloids etc.

Experiment 452. Finding the Equations of a Curve.

Let the children take a rectangular piece of paper and draw lines on it as indicated below :



The lines are parallel and

$$AE=EF=FG=GH=HD=BP=PQ=QR=RS=SC.$$

Let them fold the paper in the form of a right circular cylinder so that B, P, Q, R, S, C coincide respectively with A, E, F, G, H, D. Let them note the curve made by lines drawn on the paper. The curve is known as the right circular helix. Place the cylinder on the lattice board with its axis along the z-axis.

The children can find the coordinates of many points on the

cylinder.

The sum of x^2 and y^2 will come out to be constant ; the children may try to find another relation between x, y, z. Alternatively they may be given an iron wire in the shape of right circular helix.

In the same way, the children may be given other curves made of wires and the children may be encouraged to find the equations of the curve.

Experiment 453. Sections of a Surface.

It may be of interest for the children to take a spherical ball made of soap, say, and cut it by a knife to get various circular plane sections and similarly to take a double right circular curve and cut it by (*i*) planes perpendicular to the axis ; (*ii*) planes inclined to the axis ;

(*iii*) planes parallel to the axis ; (*iv*) planes parallel to a generator ; and (*v*) planes through the vertex.

Let the children be told about the names of these curves *viz.* circle, ellipse, hyperbola, parabola and a pair of straight lines and the importance of these curves in discussing the motions of planets and satellites.

Let them similarly consider plane sections of spheres, spheroids, ellipsoids, paraboloids, hyperboloids of one sheet, hyperboloids of two sheets etc.

16

Spherical Geometry and Spherical Astronomy

Experiments : 454 to 465

Experiment 454. Need for Study for Spherical Geometry.

In our schools, we usually study geometry of points, lines, triangles, circles etc. on a plane, but the Earth we live on is not a plane surface, it is almost a sphere. For a small neighbourhood of a point, we may regard the surface as plane, but for large areas, we have to take the curved nature of the surface of the Earth into account. When we go from one place to another, either by train or by ship or by aeroplane, we move along curved paths on a spherical surface. In these days of fast trains and jet flights, the study of spherical geometry is quite useful for the study of geography. The paths of planets, satellites, Sun, Moon and stars etc,, can be studied on the 'celestial sphere' and there also we need spherical geometry. Some knowledge of astronomy can also be given at the school level with the help of spherical geometry.

Experiment 455. Sections of a Sphere.

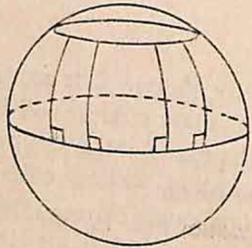
Take a hollow sphere of wood, cut it by planes. The sections will always be found to be circles. The radii of these circles will however be different. The radius will be largest if the plane of the section passes through the centre of the sphere and this radius will be equal to the radius of the sphere. The circles whose planes pass through the centre of the sphere are called great circles, others are called small circles.

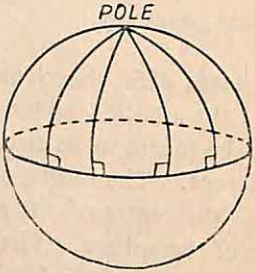
Experiment 456. 'Lines' on a Sphere.

Obviously a straight line through two points on a spherical surface does not lie on the surface. Since however we are interested in points on the surface of the sphere only, we have to modify our definition of a line. On a plane, a straight line gives the shortest distance between two points and if we stretch a string tightly between two points on a plane, the string will determine a straight line. In the same way, children may stretch strings, tightly between pair of points on the surface of a sphere, keeping the strings on the surface and making chalk marks along the strings. It will always be found that the strings always lie along great circle arcs. We may regard the great circle arcs as our 'lines' on the spherical surface. The shorter arc between two points determines the 'line segment' and the great circle passing through two points determines the line through these points. The lines on a spherical surface are finite in length as against the lines on a plane which are infinite in length. However each line here also divides the surface into two convex regions in the sense that if we join any two points in one of the regions by a line segment, the line segment lies entirely within the region and if we join two points, one in each of the regions, the line segment has to intersect the dividing line. This property is similar to that of the plane.

Through any two points on the surface, there passes a unique line lying on the surface, except when the two points are antipodals *i.e.*, when the straight line joining them passes through the centre of the sphere. In that case there is an infinity of great circles passing through two points.

Experiment 457. 'Parallel Lines' on the Spherical Surface.





From any given point, we can draw a unique 'perpendicular line' to any great circle except when the point happens to be the 'pole' of this great circle in which case, we can draw an infinite number of perpendicular lines.

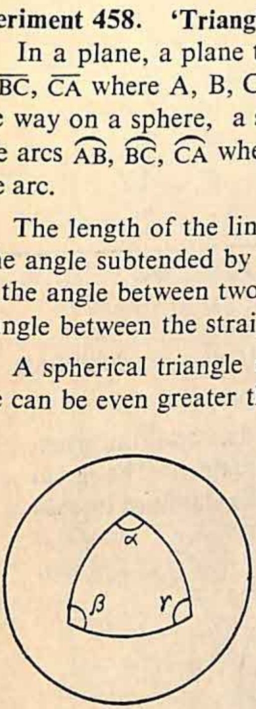
The locus of the points from which the lengths of perpendiculars to a great circle are equal is a small circle and so is not a 'line'. Each pair of great circles on the sphere intersects in two points and so there are no 'parallel' or 'non-intersecting' lines on the surface.

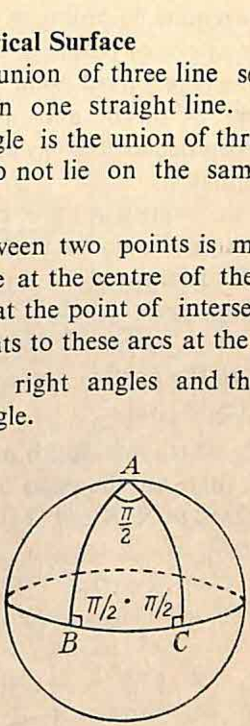
Experiment 458. 'Triangles' on a Spherical Surface

In a plane, a plane triangle is the union of three line segments $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ where A, B, C do not lie on one straight line. In the same way on a sphere, a spherical triangle is the union of three great circle arcs $\widehat{AB}$, $\widehat{BC}$, $\widehat{CA}$ where A, B, C do not lie on the same great circle arc.

The length of the line segment between two points is measured by the angle subtended by the great circle at the centre of the sphere and the angle between two great circles at the point of intersection is the angle between the straight line tangents to these arcs at the point.

A spherical triangle can have two right angles and the third angle can be even greater than a right angle.





The sum of angles of a spherical triangle is always greater than 180°. In fact the area of the spherical triangle is given by

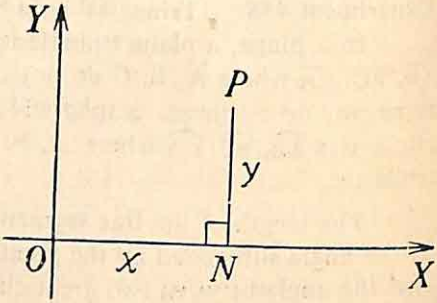
$$(\alpha+\beta+\gamma-180^\circ)\,\frac{4\pi r^2}{720}$$

where r is radius of the sphere and angles are measured in degrees.

In plane triangles, we can have triangles with same angles but with different areas. Such triangles are called 'similar triangles'. In spherical triangles on a given sphere, if the angles are equal, the areas are also equal and as such there is no theory of 'similar triangles' in spherical geometry.

We also find that Pythagoras theorem breaks down here, for in the above right-angled triangle ABC, $a = b = c = \frac{\pi}{2}$, so that $c^2 < a^2 + b^2$.

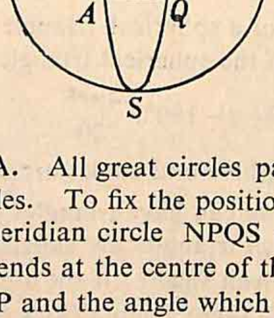
Experiment 459. Longitude and Latitude of a Point on the Surface of the Earth.



We require co-ordinates to fix the positions of points in the plane. We take a fixed point in the plane as origin and two fixed perpendicular line as axes of co-ordinates.

In the same way, we can take a fixed point on the Earth's surface as origin and two perpendicular great circle arcs through it as axes of coordinates. On the Earth's surface such coordinates are longitude and latitude respectively.

The North and South poles are antipodals and the great circle of which these are the poles is called the Equator. The great circle passing through N, S and G (Greenwich) is the standard meridian. Let



it meet the Equator in A. All great circles passing through N and S are called meridian circles. To fix the position of a point P on the surface, we draw the meridian circle NPQS through P. The angle which the arc AQ subtends at the centre of the Earth is called the longitude of the place P and the angle which the arc QP subtends at the Earth's centre is called the latitude of P. The longitude varies from 0° to 180° on the East and 0° to 180° on the West. The latitude varies from 0° to 90° North and 0° to 90° South. The meridian through N, B, and S where B is the antipodal point of A is called the date line since the date has to be changed, as ships cross this line.

The students may find the longitude and latitude of important cities of the world, like New Delhi, Calcutta, Bombay, Madras, Kanpur, London, New York, Moscow, Tokyo, Paris, Peking, Cairo, etc. by using an atlas.

They may measure shortest distances on the globe between pairs of them and may varify that the distance between two points whose longitudes are λ_1, λ_2 (both East or West) and whose latitudes are β_1, β_2 both North or South) is given by

$$r \cos^{-1} [\sin \beta_1 \sin \beta_2 + \cos \beta_1 \cos \beta_2 \cos (\lambda_1 - \lambda_2)]$$

where r is the radius of the Earth. This formula is derived in a subject known as 'Spherical Trigonometry'.

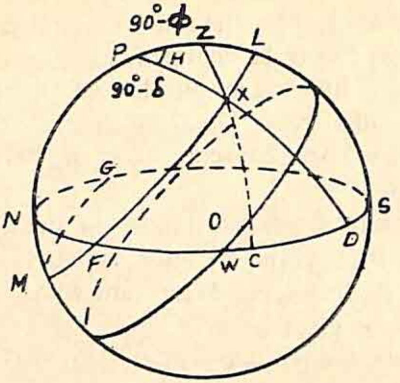
Experiment 460. Co-ordinates of Points in Space

In earlier times, we wanted to fix the position of the Sun, Moon and planets in order to study astronomy (and astrology). In modern times, we have also to fix the positions of the setallites. Ordinarily we require three coordinates to fix the position of a point in space. (In relativity theory, we require four coordinates including time, but we are not interested in this here). One coordinate can be the distance of the body from the centre of the Earth ; the other two determine the direction of the body in space. If we join the body (regarded as a point) to the Earth (also regarded as a point), the direction of the line will determine the direction of the body.

If we imagine a sphere drawn with the Earth as centre and radius unity, the above line will intersect the sphere in a point which will be called the position of the body on the celestial sphere. As the body moves, its position changes on the celestial sphere and if we know this change, we know the change in the direction of the body. The study of directions of celestial bodies is called 'Spherical Astronomy'.

In astronomy, for simplicity, we consider distance and direction separately. To fix the direction of the body, we consider the position of the body on the celestial sphere and to fix the position on the surface of the sphere we require two coordinates analogous to longitude and latitude on the surface of the Earth. We consider three such systems of co-ordinates in the next three experiments.

Experiment 461 : Declination and Hour Angle



Z is the point of the celestial sphere vertically above the observer *i.e.* it is the point where the line joining the centre of the Earth to the observer meets the celestial sphere. This point is called the observer's Zenith or simply the Zenith. Similarly the point where the line joining the centre of the Earth to the North pole of the Earth meets the celestial sphere is called the North celestial pole. The great circle through P and Z is called the observer's meridian. The great circles of which Z and P are poles are called Horizon and Celestial Equator respectively. Their points of intersection are the two cardinal points West and East respectively. The points of intersection of the observer's meridian and the horizon determine the other two cardinal points *viz* North and South.

The arc ZP = angle which arc ZP subtends at O, the centre of the celestial sphere.
= angle between OZ and OP
= $90°-\phi$

where ϕ is the latitude of the observer's station.

Let X be the position of the star at any time. Its position on the celestial sphere is determined if we know $\angle ZPX$ and the arc PX which are respectively called the Hour Angle and the North Polar Distance of the star and are usually denoted by H and $90°-\delta$ where δ is called the declination of the star. Obviously arc XD determines the declination of the star.

As a matter of fact, due to the rotation of the Earth about its axis (line OP), the star moves on a small circle LXM. The hour angle of the star depends on the time that elapses after the star was on the observer's meridian. We start measuring time from the instant the star was on the observer's meridian, and as such as the star goes from L to M, its hour angle changes from 0^h to 12^h (0° to 180°) and when it goes from M to L back, its hour angle changes from 12^h to 24^h (180° to 360°). The arc PD is 90° and the distance DX is called the declination of the star and is denoted by δ. The North polar distance is then given by $90°-\delta$, where δ may be positive or negative.

One way of fixing the position of a star on the celestial sphere is to give its declination δ and its hour angle H. Knowing H, we can draw the arc PXD and knowing δ we can find DX and then the position of X is known.

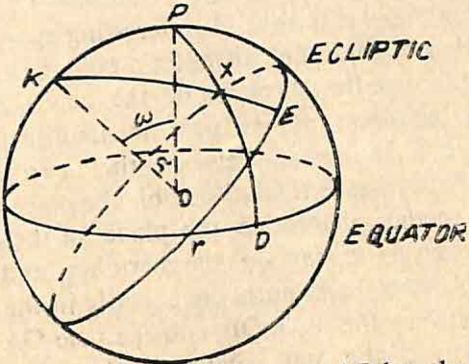
Throughout the day δ remains constant, while the hour angle changes from 0^h to 24^h.

Experiment 462. Zenith Distance and Azimuth

Another way of fixing the position of the star is in terms of $\angle PZX$ and arc ZX, known respectively as Azimuth and Zenith Distance respectively. The arc XC is called the altitude of the star and it is easily seen that it is equal to $90°-Z$. The altitude is maximum at L when the star is on the observer's meridian, it becomes 0° when the star is at F where it 'sets' and goes below the horizon. At G it 'rises' and comes above the horizon and then its altitude again begins to increase till after 24 hours it reaches L where it is said to culminate.

Experiment 463. Celestial Longitude and Latitude

The plane in which the Sun moves is called the Ecliptic and is inclined at an angle of about $23\frac{1}{2}°$ to the Equator. This inclination is called the obliquity of the ecliptic and is denoted by ϵ (epsilon) or ω (omega)



Let K be the pole of the Ecliptic and P be the pole of the Equator. OP and OK are normals to the two planes, so the angle between them is also ω. The arc PK which is equal to the angle subtended at O is also ω. The two points of intersection of the Equator and the Ecliptic are denoted by γ (the first point of Aries) and Ω (the first point of Libra). The Sun is at γ on 21st March and at Ω on 23rd September.

Let X be the position of any star. Its position is determined in terms of the arcs γD ($=\alpha$) and DX ($=\delta$) known as Right Ascension and Declination of the star respectively. The pair (α, δ) gives one set of coordinates for the star.

Similarly let the great Circle arc KX meet the Ecliptic in E, the arc γE donoted by λ is called the celestial longitude of the star and arc EX denoted by β is called the celestial latitude of the star. The pair (λ, β) gives another set of coordinates for the star.

We have thus far arrived at the following possible specifications for the position of a star on the celestial sphere :

(H, δ), $(H, 90^\circ-\delta)$; (Z, A), $(90^\circ-Z, A)$, (α, δ), (λ, β)

Different systems may be useful for different purposes and we can find relations between them through the use of spherical trigonometry.

The essential point to note is that since the surface of a sphere is two-dimensional, two coordinates are sufficient to specify the position of a point and these can be chosen in a variety of ways. Let the students think of other possible ways of choosing coordinate system on the Earth and on the celestial sphere and see why the systems we have chosen are very 'natural'.

Experiment 464. Practical Work for Children

The children will find it interesting to observe the motion of stars on a clear night. Let them fix a pole vertically in the ground. This will determine the direction of the zenith Z. Let them observe the pole star and fix another pole in its direction. This gives the direction of P. The angle between these two poles determines the angle $90^\circ-\phi$ and hence the latitude of the place of observation. The two poles together determine the plane of the observer's meridian. Let the children fix a star on the meridian and point a pole in its direction. All the three poles are initially in the observer's meridian. The angle between the pole OP and the pole OX is $90^\circ-\delta$ where δ is the declination of the star. Now as the star begins to move, the direction of OX begins to change, but the angle XOP always remains fixed so that OX describes a right circular cone with axis OP and semi-vertical angle $90° - \delta$.

At any time

(*i*) angle between OP and OZ is $90° - \phi$;

(*ii*) angle between OZ and OX gives the Zenith distance. This goes on changing;

(*iii*) angle between OP and OX gives $90° - \delta$. This remains constant;

(*iv*) angle between planes ZOP and POX determines the hour angle. After every hour this angle will increase by 15°, *i.e.*, the change in one minute of time will be $\frac{1}{4}°$. As such, this can be used as a clock at night;

(*v*) angle between planes POZ and ZOX determines the azimuth.

The above experiment is crude and it can be made more sophisticated with the help of theodolite, meridian circle etc. However it demonstrates to the children the facts that

(*a*) the pole star is practically fixed;

(*b*) all stars move towards West at a rate of 15° per hour;

(*c*) angle POX is fixed for each star and is different for different stars.

The experiment shows that all stars move from East to West at the same rate in circular paths about a certain axis pointing towards the North Pole. The only rational explanation appears to be that the Earth is rotating about its axis from West to East at this uniform rate.

In day time, the direction of OP can be fixed with the help of a compass needle and the value of the latitude of the place. Moreover, instead of motion of a star, we can observe the motion of the Sun.

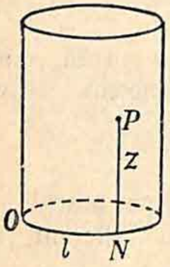
If a certain star is observed night after night, its declination will be found to be the same, but if the Sun is observed throughout the year, its declination will be found to change significantly. It will be found to be 0° on 21st March (vernal equinox) and on 23rd September (autumonal equinox), it will be $23\frac{1}{2}°$ on 21st June (summer solistice) and $-23\frac{1}{2}°$ on 22nd December (winter solistice). A similar situation applies to planets.

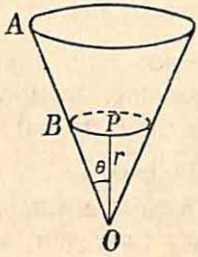
The students can also visit planetariums and get some ideas about motion of stars.

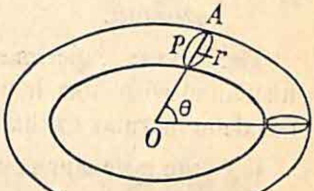
They can also take large globes, measure distance betwen cities by stretching strings. They may be asked to find the shortest ocean routes between ports and then compare them with the actual ones used. Similarly they may find the shortest air routes between two cities when the plane has to touch at some other cities on the way, calculate the distances and compare with the actual distances and routes.

Experiment 465. Geometry on other Surfaces

Plane and spherical geometries are practically important, but the students can study in the same manner geometries on other surfaces. In particular they can study plane sections and fixing of coordinate systems.







On a cylinder, they can fix a point, take a section through this point. The coordinates for P can be the arc length ON and the vertical distance Z. On a right circular cone, a generator OA can be fixed and the coordinates for P can be the distance PO and the arc length PB. On a torus, we can take a fixed circular section and the co-ordinates of any point P are the angle ϕ which OP makes with OA and the angle which the plane of the circular section in which P lies makes with a fixed plane circular section. (Here O is centre of the circular section in which P lies and OA is in a fixed direction in this section).

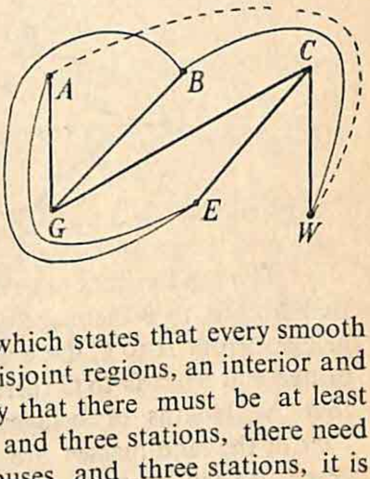
17

Networks and Polyhedra

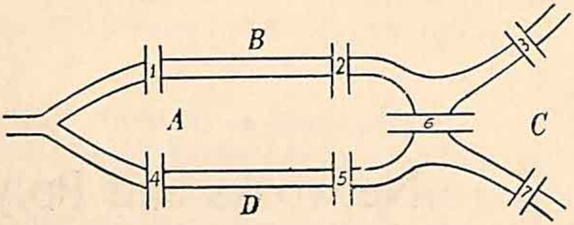
Experiments : 466 to 475

Experiment 466. The Three Utilities Problem.

There are three houses A, B, C and three stations G, E, W supplying gas, electricity and water. We may want to provide pipes in one plane connecting each of the stations with all the houses and we do not want them to intersect or we may want to construct non-intersecting roads joining each of houses A, B, C to each of stations G, E, W. Is it possible? However hard we may try, we do not succeed. Of course this does not prove that it will not be possible. However by using Jordan's closed curve theorem which states that every smooth plane curve divides the plane into two disjoint regions, an interior and an exterior, we can prove mathematically that there must be at least one intersection. If we had two houses and three stations, there need not be an intersection. With four houses and three stations, it is easily seen that there must be at least three intersections and with four houses and four stations, the least number of intersections is five. However on an inflated cycle tube or torus, it is possible to connect G, E, W with A, B, C without any intersection. The students may verify it and they may also try to verify statements about the other problems mentioned here.

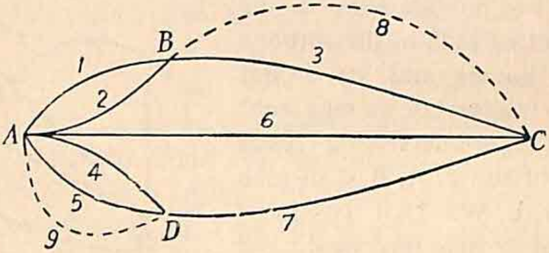


Experiment 467. Koinsberg Seven Bridges Problem.



A, B, C, D are four land masses and there are seven bridges on the river which flows through the town. The problem is to start from any point, cross all the bridges once and once only and return to the starting point. People tried to find this path by actually walking in the evenings over the seven bridges in the town of Koinsberg for years and did not succeed until Euler proved that it is impossible.

The first essential step is to reduce it to the following equivalent problem.



The land masses are reduced to points A, B, C, D and we find that from A to B there are two bridges, from A to D there are two bridges, from A to C there is one bridge, from B to C there is one bridge and from D to C there is one bridge. It is important to realise that the lengths of bridges, their being straight or curved and the sizes of the land masses are irrelevant to the problem. If in this graph, we can start from a point, traverse each of the edges once and once only and return to the original point, there will be a corresponding solution of the original seven bridges problem. Instead of walking on the bridges, one can take a pencil and try whether the above graph can be traced without lifting the pencil from the paper. A trial which would have taken one hour of walking can be done in one second on paper.

The graph has one vertex *viz.*, A of degree 5 (*i.e.*, five edges meet at this point) and three vertices *viz.*, B, C, D of degree 3 each. If it were possible to start from A and return to A by a different bridge, there must be an even number of edges meeting at A. In our case all the vertices are of odd degree and therefore it is not possible to start from any vertex and return to the same vertex after moving over each edge once only.

If however our condition is not to return to the original point, but to cross the bridges once and once only, two of the vertices can be of odd degree, one is the vertex where the path starts and the other is the vertex at which the path ends. If more than two vertices are of odd degree, even this will not be possible. Thus suppose there is an additional bridge *viz*., 8 from B to C (shown by dotted line above) we could have started from A crossed bridges 1, 2, 3, 4, 5, 6, 7 and 8 in this order and the journey would end at D. Other paths are also possible.

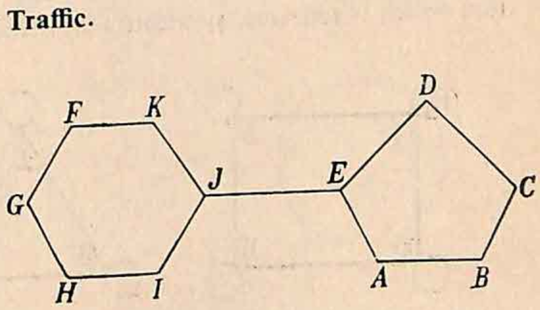
If there was still another bridge *viz*., 9 from A to D, then all the vertices would be of even degree and we can have a path starting from A and crossing bridges in the order 1, 2, 4, 5, 6, 3, 8, 7 and 9 and returning to A. Other paths are also possible.

Euler showed that a simply connected network may be completely traversed in a single journey if and only if there are either no vertices or there are just two vertices at which odd number of edges meet.

The students can draw a large number of networks with no odd vertices, two odd vertices, more than two odd vertices and verify Euler's result.

Experiment 468. One-way Traffic.

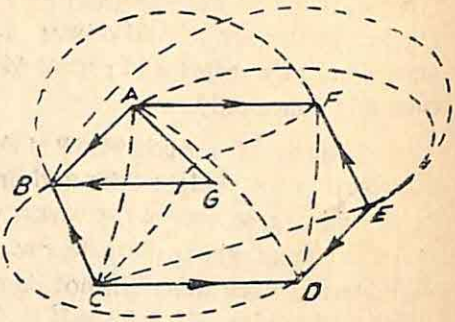
Here is a city road map. Can we introduce one-way traffic without disconnecting some paths of the town? It is obvious that on EJ traffic has to be allowed both ways.



The students can attempt to find other possible paths along which one-way traffic is not allowed and they can deduce some general results.

Experiment 469. League Matches

A, B, C, D, E, F and G are seven teams playing a league tournament wherein each team has to play against every other team. The network shows the position on any date. It shows that team A has beaten teams B, F and G, team C has beaten teams B and D, team E has beaten teams D and F, team G has beaten B. The graph also shows the matches yet to be played. (These are shown by dotted lines).

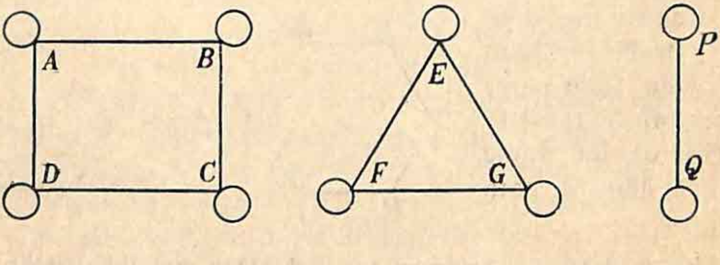


Experiment 470. Graphs and Relations

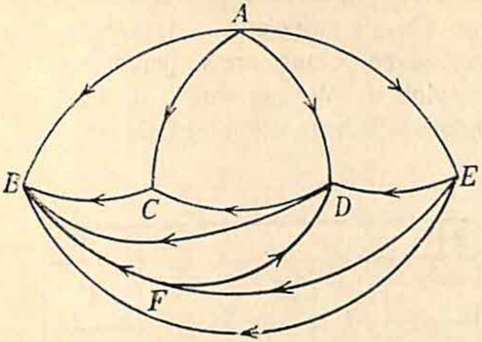
The elements of a set are denoted by vertices of a graph and we draw a directed edge from A to B if A is related to B. If A is related to B and B is related to A, the edge has to be directed both ways and we keep it undirected to indicate that in fact it is directed both ways.

If the relation is reflexive, every element is related to itself and we get loops from each vertex to itself. If the relation is symmetric, every edge is undirected. If the relation is transitive, then whenever there is a directed edge from A to B and a directed edge from B to C, then there has to be a directed edge from A to C.

The following graph shows an equivalence relation *i.e.*, a relation which is reflexive, symmetric and transitive.

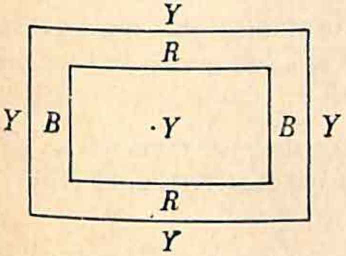


We also see the equivalence classes here. The following graph shows a relation which is transitive but is not reflexive or symmetric.

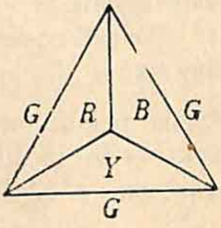


Experiment 471. Map-colouring Problems.

How many colours are necessary to colour a map in such a way that regions or countries which meet along an edge or boundary are coloured differently ?

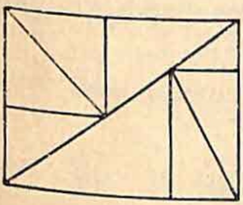


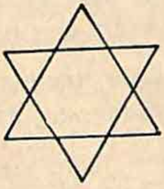
3 colours are required

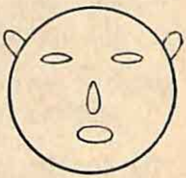


4 colours are required

Let the children draw a large number of possible maps and verify that four colours are always sufficient.

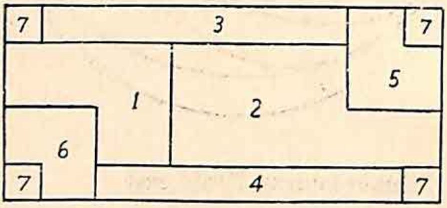






Similarly let them draw maps on spheres. How many colours are required here ?

It has been found that for colouring any map on a plane or a sphere, four colours are sufficient, though the result has been 'proved' only for maps with 38 regions or less. It has however been proved that five colours are always sufficient, whatever be the number of regions. For a torus, seven colours are sufficient *e.g.*, in the following map, we use seven colours. We can fold it in the form of a torus and contiguous regions will have different colours.



Experiment 472. Construction of Polyhedra with the Help of Sockets and Pencils.

In experiments 287—290, we have studied the construction of polyhedra. We make some additional remarks in this and the following experiments :

A regular polyhedron has the same shape at every vertex. There are the same number of edges meeting at each vertex and the angle between every pair of edges is the same.

(*i*) For the cube, there are three edges at each vertex and the angle between every pair is a right angle.

(*ii*) For the tetrahedron, there are three edges at each vertex and the angle between every pair is 60°.

(*iii*) For the octahedron, there are four edges at each vertex and the angle between every pair is again 60°.

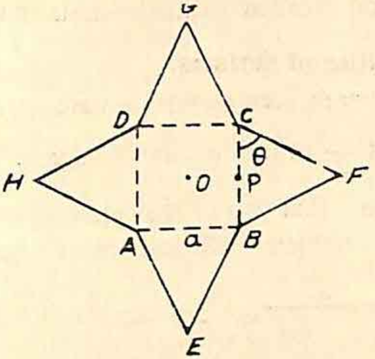
(*iv*) For the dodacahedron, there are three edges at each vertex and the angle between every pair is 108°.

(*v*) For the icosahedron, there are five edges at each vertex and the angle between every pair is 60°.

Separate plastic sockets can be made for each regular polyhedron and by fitting in 'pencils' into these edges to represent the edges we can easily make the regular polyhedra.

Experiment 473. Restrictions on the Lengths of Edges of Polyhedra.

Suppose we have to construct a square pyramid. The net is shown below.



The base is the square ABCD and the other faces are in the form of isosceles triangle. On folding along the dotted lines, we get the square pyramid and FOP becomes a right-angled triangle in which by Pythagoras theorem

$$FP^2 = FO^2 + OP^2$$

$$FP^2 > OP^2$$

or $$(CF)^2 - (CP)^2 > (OP)^2$$

or $$(CF)^2 > (\tfrac{1}{2}a)^2 + (\tfrac{1}{2}a)^2$$

$$CF > \frac{a}{\sqrt{2}}$$

$$\cos\theta = \frac{CP}{CF} < \frac{\frac{a}{2}}{\frac{a}{\sqrt{2}}} = \frac{1}{\sqrt{2}}$$

$\therefore$ $$\theta > 45^\circ$$

We can find similar conditions for other pyramids.

Experiment 474. Construction of a Rhombo-Dodacahedron.

Construct square pyramids with height equal to half the side of the square base. For this

$$(FP)^2 = (FO)^2 + (OP)^2 = \left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2 = \frac{a^2}{2}$$

$$(CF)^2 = (FP)^2 + (PC)^2 = \frac{a^2}{2} + \left(\frac{a}{2}\right)^2 = \frac{3}{4}a^2$$

$$CF = \tfrac{3}{2}a$$

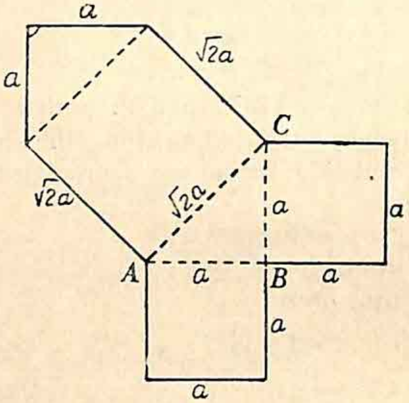
Construct six such pyramids and place these inside a cube of side a. These will fit exactly.

Now fix these six square pyramids on the outside of the six faces of the cube. We get a polyhedron with 12 faces, each being a rhombus. Thus polyhedron is called a rhombo-dodacahedron.

Experiment 475. Fitting of Surfaces.

(*i*) We have already seen how six square pyramids of edge a and height $\frac{a}{2}$ fit into a cube of edge a.

(*ii*) We can check that two of the prisms whose net is given below also completely fill the cube of edge a.



(*iii*) We can construct a tetrahedron with edge equal to the diagonal of the face of a cube. When fixed inside the cube, the six edges of the tetrahedron will coincide with the six diagonals of the faces of the cube.

We conclude this section with some remarks on the advantages of the study of polyhedra.

(*i*) Solid bodies are certainly less abstract than plane figures. The students can 'handle' three-dimensional objects in a literal sense. They get the feeling of creativity to a greater degree here.

(*ii*) When a student who has studied two-dimensional plane geometry exclusively for four or five years, begins the study of three-dimensional objects, he meets with certain difficulties since in a certain sense his intuition and imagination have become completely oriented to thinking in terms of plane figures. A student who starts with a combined study of two and three dimensional geometry faces no such difficulty.

(*iii*) For some students, properties of polyhedra may provide many interesting results which may be more satisfying than the results for plane figures only.

(*iv*) Students are interested in the concepts of fitting which is a motor sensation. Students find how edges of nets match one another, how prisms and pyramids fill a cube and how the square pyramids pasted on the cube faces merge pairwise in each other ; this can be felt by them by stroking along the surfaces. At a young age, motion component of personality is strongly marked and young students love learning by doing things with their hands.

(*v*) The study of mathematics requires development of imagination. The study of polyhedra helps in this process.

(*vi*) The study of polyhedra can lead to the study of areas and volumes.

(*vii*) The students can make plasticine models of polyhedra, can deform them, stretch them without tearing and verify that Eulers formula continues to hold. This can provide an introduction to topology or rubber-sheet geometry.

18

Probability

Experiments : 476 to 479

Experiment 476. Definition of Probability.

(*i*) Consider the throw of a die. There are six possibilities in the sense that 1 or 2 or 3 or 4 or 5 or 6 may turn up. On the number line, we get six points.

These six points constitute our *sample space*. How many of these numbers are odd ? The answer is '3'. Thus three out of six points of sample space are odd. This is expressed by saying that the probability of the number turned up being odd is $\frac{3}{6}$ or $\frac{1}{2}$.

How many of these numbers are multiples of 3 ? The answer is '2', and so the probability of the number turned up being a multiple of 3 is $\frac{2}{6}=\frac{1}{3}$.

How many of these are multiples of 4 ? The probability of the number turned up being a multiple of 4 is $\frac{1}{6}$.

(*ii*) Now consider the throw of two dice and note the number occuring on each die. We get 36 possible pairs which are represented in the following sample space.

6	*	*	*	*	*	*
5	*	*	*	*	*	*
4	*	*	*	*	*	*
3	*	*	*	*	*	*
2	*	*	*	*	*	*
1	*	*	*	*	*	*
	1	2	3	4	5	6

There are 36 points in the sample space. How many give the sum as 7 ? These are (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1). The probability of the sum of points being 7 is $\frac{6}{36}$. Which ones give the sum as 10 ? These are

(4, 6), (5, 5), (6, 4)

The probability of the sum of points being 10 is $\frac{3}{36}$. How many of these give sum of numbers to be even ? These are :

(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (3, 6)
(3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (4, 6)
(5, 1), (5, 3), (5, 5), (6, 2), (6, 4), (6, 6)

The probability of the sum being even is $\frac{18}{36}$.
Let the children answer questions like the following :
What is the probability that the sum is a perfect square ?
What is the probability that the sum is a cube number ?
What is the probability that the sum is a prime number ?
What is the probability that the product is a prime number ?
What is the probability that the product is a perfect square ?
The children should be encouraged to ask questions of this type themselves and to answer them.

(*iii*) With three dice, we shall get a three dimensional sample space with 216 points.

For how many of these points well the sum be 7 ? Such points are :

(1, 1, 5), (1, 2, 4), (1, 3, 3), (1, 4, 2), (1, 5, 1),
(2, 1, 4), (2, 2, 3), (2, 3, 2), (2, 4, 1),
(3, 1, 3), (3, 2, 2), (3, 3, 1),
(4, 1, 2), (4, 2, 1),
(5, 1, 1).

The probability of the sum being 7 with three dice is $\frac{15}{216}$. Let the children ask and answer questions similar to the above.

(*iv*) The 52 cards of a pack give 52 points in a sample space.

clubs	*	*	*	*	*		*	*	*	*	*	*	*
spades	*	*	*	*	*	*	*	*	*	*	*	*	*
diamonds	*	*	*	*	*	*	*	*	*	*	*	*	*
hearts	*	*	*	*	*	*	*	*	*	*	*	*	*
	Ace	2	3	4	5	6	7	8	9	10	J	Q	K

What is the probability that a card is a king ?	$\frac{4}{52}$
What is the probability that the card is a club ?	$\frac{13}{52}$
What is the probability that the card is red ?	$\frac{26}{52}$

Experiment 477. Relative Frequency.

(*i*) Let each child take a coin, throw it a number of times and note his result as follows. When he gets a head, he makes a mark 1 on the paper. After 4 such marks, when he gets a fifth head, he crosses the four marks he has made. Thus suppose in 30 throws, he gets 12 heads, then his record will appear as follows :

Let each child prepare a table as follows :

Number of throws	20	40	50	80	100	120	150	180	200
Number of heads	...	...	...	...	...	...	...	...	...
$r=\frac{\text{Number of heads}}{\text{Number of throws}}$	...	...	...	...	...	...	...	...	...
$\lvert r-\frac{1}{2} \rvert$	...	...	...	...	...	...	...	...	...

Let the children note whether $\lvert r-\frac{1}{2} \rvert$ becomes smaller and smaller as the number of throws increases.

(*ii*) Let each child throw a die and note the number of times he gets 1 or 2 or 3 or 4 or 5 or 6. Let him keep his record as follows :

Number of throws :	60	120	180	240	300
Number of times he gets 1 :	...	...	...	...	...
Number of times he gets 2 :	...	...	...	...	...
... ...	...	...	...	...	...
Number of times he gets 6 :	...	...	...	...	...

$r_1=\frac{\text{No. of times he gets 1}}{\text{No. of throws}}$ =

$r_2=\frac{\text{No. of times he gets 2}}{\text{No. of throws}}$ =

... ...

$r_6=\frac{\text{No. of times he gets 6}}{\text{No. of throws}}$ =

$\lvert r_1-\frac{1}{6} \rvert$ =

$\lvert r_2-\frac{1}{6} \rvert$ =

... ...

... ...

$\lvert r_6-\frac{1}{6} \rvert$ =

What conclusion does each child draw ? What is the connection between probability of getting a number and the corresponding ratio obtained above ? This ratio is called *relative frequency* for the given number.

(*iii*) Let two children throw a die each and let a third child note when the sum of points is 7. Let him record his observation as follows :

No. of throws	36	72	108	144	180
No. of times the sum is 7	...	...	...	...	...
$r=\frac{\text{No. of times the sum is 7}}{\text{No. of throws}}$	...	...	...	...	...
$\mid r-\frac{1}{6}\mid$					

What conclusion can be drawn ?

(*iv*) Let three children throw one die each and let a fourth child note when the sum of points is 7. Let him note as follows :

No. of throws :	36	72	108	144	180
No. of times the sum is 7	...	...	...	...	...
$r=\frac{\text{No. of times the sum is 7}}{\text{No. of throws}}$	...	...	...	...	...
$\mid r-\frac{5}{36}\mid$	...	...	...	...	...

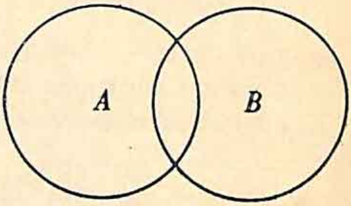
What conclusion can be drawn ?

(*v*) Let one child shuffle well a pack of cards and let another child draw a card and note whether it is a heart and then replace it. The first child reshuffles again and the second child draws another card and notes whether it is a heart. This process, can be continued a number of times.

Number of times a card is drawn :	20	40	60	80	100
Number of times it is a heart :	...	...	...	...	...
ratio r :	...	...	...	...	...
$\mid r-\frac{1}{4}\mid$:	...	...	...	...	...

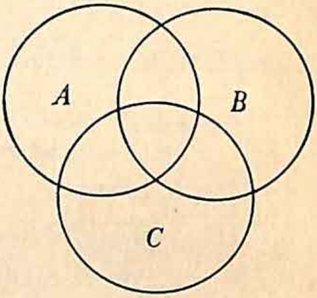
In all the five cases, it will be found that as the number of draws increases, the ratio giving the relative frequency comes nearer and nearer to the value obtained as probability in the last experiment. Does it always so happen ? What conclusion do the children draw ?

Experiment 478. Use of Venn Diagrams to Solve some Combinatorial Problems.



Let $n(A)$, $n(B)$, $n(A \cup B)$, $n(A \cap B)$ denote the number of elements in the corresponding sets, so that $n(A \cup B)$ denotes the number of elements in $A \cup B$. If we find $n(A)+n(B)$, we will count all the elements of $A \cup B$, but the elements of $A \cap B$ will be counted twice so that we get :

$$n(A \cup B)=n(A)+n(B)-n(A \cap B)$$



Similarly to get $n(A \cup B \cup C)$, we consider $n(A)+n(B)+n(C)$. In this latter sum, we count elements of $A \cap B$, $B \cap C$, $C \cap A$ twice, so we subtract the number of these elements once, but in this subtraction we have subtracted the number of elements of $A \cap B \cap C$ twice, so we add this number once to get

$$\begin{aligned} n(A \cup B \cup C)=n(A)+n(B)+n(C)-n(A \cap B) \\ -n(B \cap C)-n(C \cap A)+n(A \cap B \cap C) \end{aligned}$$

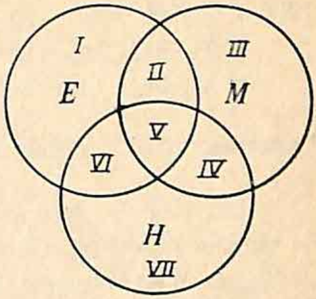
Arguing in the same way, we get

$$\begin{aligned} n(A \cup B \cup C \cup D)=\ & n(A)+n(B)+n(C)+n(D) \\ & -n(A \cap B)-n(A \cap C)-n(A \cap D)-n(B \cap C) \\ & -n(B \cap D)-n(C \cap D) \\ & +n(A \cap B \cap C)+n(A \cap B \cap D)+n(A \cap C \cap D) \\ & +n(B \cap C \cap D)-n(A \cap B \cap C \cap D) \end{aligned}$$

We can use these formulae or the Venn diagrams to solve some combinatorial problems *e.g.*, suppose we are given the following data from a class of 70 students :

Number of those who took economics only = 6
Number of those who took both economics and mathematics = 36
Number of those who took mathematics = 50
Number of those who took economics, but not history = 18
Number of those who took economics = 53
Number of those who took history and mathematics = 34

What conclusions can we draw from this data ?



We divide the Venn diagram into seven disjoint regions, then the given data gives :

$$\begin{aligned}
&n(\mathrm{I})+n(\mathrm{II})+n(\mathrm{III})+n(\mathrm{IV})+n(\mathrm{V})+n(\mathrm{VI})+n(\mathrm{VII})=70\\
&n(\mathrm{I})=6\\
&n(\mathrm{II})+n(\mathrm{V})=36\\
&n(\mathrm{II})+n(\mathrm{III})+n(\mathrm{IV})+n(\mathrm{V})=50\\
&n(\mathrm{I})+n(\mathrm{II})=18\\
&n(\mathrm{I})+n(\mathrm{II})+n(\mathrm{V})+n(\mathrm{VI})=53\\
&n(\mathrm{IV})+n(\mathrm{V})=34
\end{aligned}$$

From these equations we get

$n(\mathrm{I})=6$, $n(\mathrm{II})=12$, $n(\mathrm{III})=4$, $n(\mathrm{IV})=10$, $n(\mathrm{VI})=11$, $n(\mathrm{VII})=3$

From these we can easily deduce conclusions like the following :

Number of students taking all three subjects $=n(\mathrm{V})=24$

Number of students taking economics and history $=n(\mathrm{V})+n(\mathrm{VI})=35$

Number of students taking history but not economics and mathematics $=n(\mathrm{VIII})=3$

We can also use Venn diagrams to test the consistency of any data. If solution gives a negative number in any region, the data are inconsistent. Thus suppose the following data are given for 1000 employees on a cotton mill as to their race, sex and martial status : 525 coloured lives, 312 male lives, 470 married lives, 42 coloured males, 147 married coloured, 86 married males, 25 married coloured males.

Let M stand for male, C for coloured and W for married status then

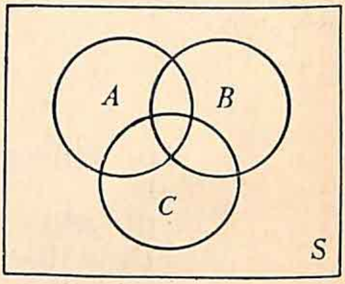
$$\begin{aligned}
n(M\cup C\cup W)&=n(M)+n(C)+n(W)-n(M\cap C)-n(M\cap W)\\
&\quad -n(C\cap W)+n(M\cap C\cap W)\\
&=312+525+470-42-86-147+25\\
&=1307-275+25=1057
\end{aligned}$$

But we are given $n(M\cup C\cup W)=1000$. Thus the data given are inconsistent.

We can also deduce many inequalities from the above relation, *e.g.*,

$$n(A\cup B\cup C)\leqslant n(A)+n(B)+n(C)$$
$$n(A\cup B\cup C)\geqslant n(A)+n(B)+n(C)-n(A\cap B)-n(B\cap C)-n(C\cap A)$$

Experiment 479. Theorems of Probability.



Let S be the sample space so that its various points represent all possible outcomes of an experiment. We consider only sample spaces with a finite number of outcomes. A, B, C are subsets of S. We define *probability* of A, B, C respectively by

$$P(A)=\frac{n(A)}{n(S)},\ P(B)=\frac{n(B)}{n(S)},\ P(C)=\frac{n(C)}{n(S)}$$

we then get

$$P(A\cup B)=\frac{n(A\cup B)}{n(S)},\ P(A\cap B)=\frac{n(A\cap B)}{n(S)}$$

A, B, C are called *events*.

From the equations (I) and (II) of the last experiment, by dividing throughout by $n(S)$, we get

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$
$$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)$$
$$-P(C\cap A)+P(A\cap B\cap C)$$

This is called the *theorem of total probability*.

If A, B, C are disjoint sets *i.e.*, if $A\cap B=B\cap C=C\cap A=\phi$, then since $P(\phi)=0$, we get

$$P(A\cup B)=P(A)+P(B)$$
$$P(A\cup B\cup C)=P(A)+P(B)+P(C)$$

In this case no outcome can be common to events A, B, C and no two of these events can occur together. We say A, B, C are *mutually exclusive events*, since the happening of one implies that the others do not happen. The probability of the happening of any one of a number of mutually exclusive events is the sum of the probabilities of these events.

Now the points of $A\cap B$ are common to both A and B and so an outcome lying in $A\cap B$ indicates that both events A and B have happened so that $P(A\cap B)$ denotes the probability of happening of A and B.

$$P(A\cap B)=\frac{n(A\cap B)}{n(S)}=\frac{n(A\cap B)}{n(A)}\times\frac{n(A)}{n(S)}$$
$$=P(A)\frac{n(A\cap B)}{n(A)}$$

What can be the interpretation of the second factor? If we confine our attention to set A only *i.e.*, if we consider this as our sample space, then the second factor gives the conditional probability for $A\cap B$, given that event A has occurred. We denote this by P(B, A) and call it the *conditional probability* of B to happen when A is known to have happened. Thus

$$P(A\cap B)=P(A)\,P(B, A)$$

This is known as the *theorem of compound probability*.

Two events A and B are said to be *independent* if the happening of B does not depend on the happening of A. For such events $P(B, A)=P(B)$, so that in this case the theorem of compound probability gives

$$P(A\cap B)=P(A)P(B)$$

Thus the probability of the happening of two independent events together is the product of their probabilities. Consider now the following example.

(*i*) A box contains a white and b black balls. The sample space has $a+b$ points of which a points correspond to a white and b points correspond to b black balls. The probability of a white ball being drawn is therefore $\frac{a}{(a+b)}$.

(*ii*) Suppose this ball is white and it is put aside. The sample space has $a+b-1$ points of which $a-1$ correspond to white balls. The probability of the second ball being white is $\frac{(a-1)}{(a+b-1)}$.

(*iii*) On the other hand if the first ball is black, the probability of the second ball being white is $\frac{a}{(a+b-1)}$.

(*iv*) If the first ball is replaced before the second is drawn, probability of the second ball being white is $\frac{a}{(a+b)}$ and this is independent of what the colour of the first ball was.

(*v*) If the colour of the first ball is not noticed, there are two possibilities for the second ball to be white.

(1) The first ball is white, the second ball is white.

(2) The first ball is black, the second ball is white.

They are mutually exclusive and so their probabilities can be added.

By using the theorem of compound probability, the probability of the first possibility is $\frac{a}{a+b}\times\frac{a-1}{a+b-1}$ and the probability of the second possibility is $\frac{a}{a+b}\times\frac{b}{a+b-1}$.

(*vi*) By using the theorem of total probability, the probability of the second ball being white is the sum of these two and the sum is $\frac{a}{(a+b)}$.

Let the children do more such examples.

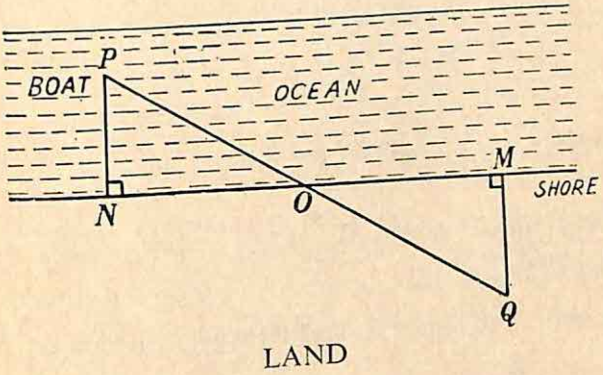
19

Patterns in Surveying

Experiments : 480 to 484

Some examples in surveying require knowledge of elementary geometry only and can easily demonstrate the power of mathematics to the school child.

Experiment 480 To Find the Distance of a Boat in the Ocean from the Shore without Getting into the Ocean.



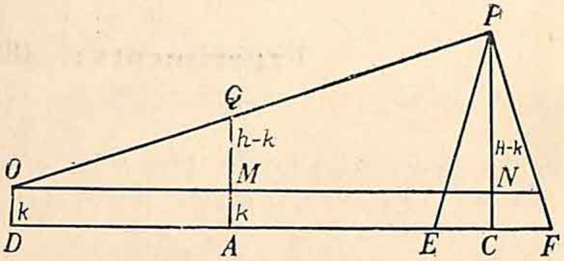
We have to find the distance NP. From N move along the shore (*i.e.*, along a line perpendicular to NP) a certain distance NO and fix a pole there. From O move to a point M in the same direction where OM=ON. From M move inland at right angles to NM and go on moving till you reach a point Q which is such that Q, the pole at O and the boat are in the same straight line.

From congruent triangles ONP and OMQ, NP=MQ so that QM is easily measured and hence the distance NP is known.

Experiment 481. To Find the Height of a Tower without Going up the Tower.

The tower may be the Kutab Minar or the leaning tower of Pisa or one of the Egyptian pyramids or the Effil tower or may be any high building.

Fix a pole of known height h in the ground at some distance from the tower and move backwards till you find your eye, the top of the pole and the top of the tower in the same straight line.



Let k be the height of the eye above the ground.

From similar triangles QMO and PNO, we get

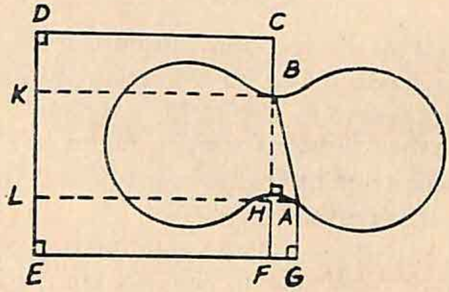
$$\frac{PN}{ON}=\frac{QM}{OM}$$

or $$\frac{H-k}{ON}=\frac{h-k}{OM}$$

or $$\frac{H-k}{\frac{DE+DF}{2}}=\frac{h-k}{DA}$$

All the quantities except H can be directly measured and so H can be determined.

Experiment 482. Building a Tunnel through a Mountain.



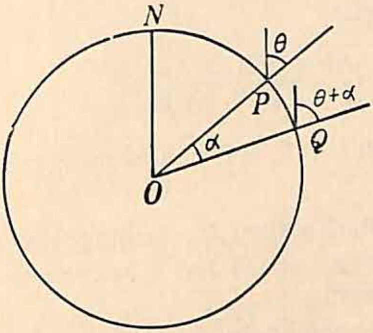
We have to dig a straight tunnel from A to B, possibly digging from both ends simultaneously. We cannot, of course, possibly see through the mountain.

We have to determine the direction of digging at A or B.

$$\frac{BH}{AH}=\frac{KL}{FG}=\frac{DE-DK-EL}{EG-EF}=\frac{DE-BC-AG}{EG-CD}$$

All the quantities on the R.H.S. can be measured and therefore $\frac{BH}{AH}$ can be found, giving the direction of digging at both ends.

Experiment 483. Measuring the Size of the Earth.



Take any point on the surface of the Earth and move towards the South a certain distance say a mile to reach Q. At P, find the angle θ between the direction of the plumb line (giving the direction of OP) and the direction of the Pole star. At Q, find the angle $\theta+\alpha$ between the direction of the Pole star and the direction of the plumb line. Then

$$\angle POQ=\alpha$$

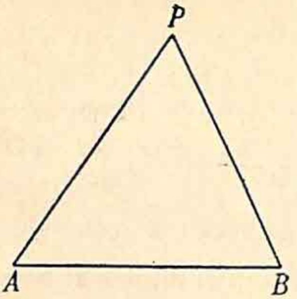
If α is measured in degrees, we have

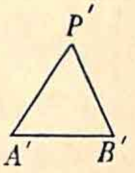
$$\frac{\alpha}{360}=\frac{\text{Length PQ}}{\text{Circumference of the Earth}}$$

Thus circumference of the Earth is known. Dividing this by 2π, we get the radius of the Earth.

Experiment 484. Finding Distance of an Inaccessible Object.

Let P be the inaccessible object. We choose a base line AB and measure angles BAP and ABP with the help of a theodolite. The



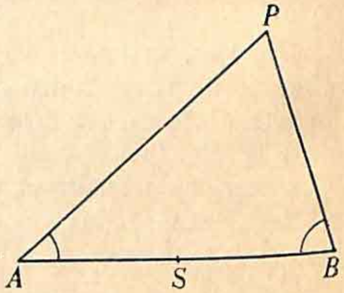


third angle BPA is then easily known. We construct a triangle with there three angles with any arbitrary base and measure its three sides. From similar triangles,

$$\frac{AP}{A'P'}=\frac{BP}{B'P'}=\frac{AB}{A'B'}$$

Since AB, A'B', A'P', B'P' can be measured, we are able to fin AP and BP.

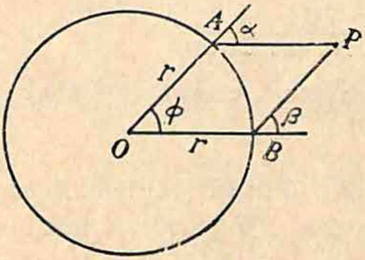
The same method is used for finding the distances of distant stars. Let P be the star and A and B be the positions of the Earth at an interval of six months.

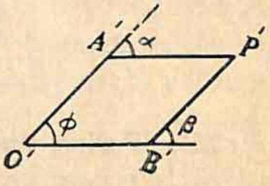


If S be the position of the Sun, then AS=SB=93 million miles.

Measurement of PAB and PBA and knowledge of AB, determine PA and PB.

For measuring the distances of planets and Moon, we choose two points A and B on the Earth on the same meridian *i.e.*, having same longitude, but different latitudes.





We measure angles a, β at A and B. These are the angles which the directions of AP and BP make with the vertical directions at the two places. The angle AOB is the difference in the latitudes of the two places. We can now easily construct on paper a figure similar to OBPA, then

$$\frac{PA}{P'A'}=r, \quad \frac{PB}{P'B'}=r$$

and knowing the radius of the Earth, we can easily find PA and PB.

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Some Applications of Mathematics

Experiments : 485 to 489

Experiment 485. The Sequencing Problem.

Suppose there are five jobs 1, 2, 3, 4, 5 to be done on the two machines A and B in the order AB. The times taken on the two machines are given by

Job	Processing time on machine A in hours	Processing time on machine B in hours
1	5	2
2	1	6
3	9	7
4	3	8
5	10	4

The jobs may be done in any order. We have to find that order or sequence which takes the minimum time. There are $\underline{|5} = 120$ possible permutations or sequences and we can find the time taken for each of these *e.g.*, for the natural sequence 1, 2, 3, 4, 5 we have

	Machine A		Machine B	
Job	Time in	Time out	Time in	Time out
1	0	5	5	7
2	5	6	7	13
3	6	15	15	22
4	15	18	22	30
5	18	28	30	34

Time taken is 34 hours, machine A is idle for 6 hours and machine B is idle for 7 hours.

The children will find it a great fun to try to find the times taken by other arrangements *e.g.*, for the sequence 3, 5, 4, 2, 1 they get

	Machine A		Machine B	
Job	Time in	Time out	Time in	Time out
3	0	9	9	16
5	9	19	19	23
4	19	22	23	31
2	22	23	31	37
1	23	28	37	39

This arrangement is worse, since the total time is now 39 hours, machine A is idle for 11 hours and machine B is idle for 12 hours.

To find the best arrangement, we note that in the beginning machine B is idle and that at the end machine A is idle. Let us choose the arrangement so that these times are least. We find the least processing time to be 1 hour for job 2 by machine A, so we process the job 2 first. For the remaining 4 jobs, the least processing time is 2 hours on machine B. We choose the job last so that the arrangement obtained is

2				1

For the remaining three jobs, the times are :

Job	Time on machine A	Time on machine B
3	9	7
4	3	8
5	10	4

Since job 4 takes least time on machine A, we choose it first and since job 5 takes least time on machine B, we choose it last giving the final arrangement.

2	4	3	5	1

For the arrangement, we have

	Machine A		Machine B	
Job	Time in	Time out	Time in	Time out
2	0	1	1	7
4	1	4	7	15
3	4	13	15	22
5	13	23	23	27
1	23	28	28	30

The time taken is only 30 hours, machine A is idle for 2 hours only and machine B is idle for 3 hours only.

We have obtained above the following rule which has to be justified rigorously mathematically.

Let the processing times on machine A be $A_1, A_2, \ldots, A_n$ and on machine B, let these be $B_1, B_2, \ldots, B_n$.

(*i*) Select the smallest processing time on either machine. If there is a tie, choose either of them.
(*ii*) If the selected time is A_r do the *r*th job first, if the selected time is B_s, do the *s*th job last.
(*iii*) There are $n-1$ more jobs more to be done. Apply the same rule to the reduced set of processing times, obtained by deleting the machines processing times for the job assigned already.
(*iv*) Continue this process till all the jobs are assigned.

Experiment 486. The Transportation Problem.

Stations A_1, A_2, A_3, A_4 have 5, 6, 2, 9 wagons available and stations $B_1, B_2, B_3, B_4, B_5, B_6$ require 4, 4, 6, 2, 4, 2 wagons. The total number of wagons required is equal to the total number of wagons available. Our first problem is to give some feasible solutions.

One method is as follows. We satisfy the requirements of B_1 from A_1 if possible. Any surplus is given to B_2 and the remaining needs of B_2 are met from A_2 and so on, so that we get

	B_1	B_2	B_3	B_4	B_5	B_6	Available
A_1	4	1					5
A_2		3	3				6
A_3			2				2
A_4			1	2	4	2	9
Required	4	4	6	2	4	2	22

By permuting $B_1, B_2, \ldots, B_6$, we get $|\underline{6}$ feasible solution. By permuting A_1, A_2, A_3, A_4 also we get a total of $|\underline{6} \times |\underline{4}$ feasible solutions. Thus we get another feasible solution.

	B$_3$	B$_1$	B$_2$	B$_5$	B$_4$	B$_6$	
A$_1$	5						5
A$_2$	1	4	1				6
A$_3$			2				2
A$_4$			1	4	2	2	9
	6	4	4	4	2	2	22

These 17280 are not the only feasible solutions *e.g.*, the following is a different feasible solution :

	B$_1$	B$_2$	B$_3$	B$_4$	B$_5$	B$_6$	
A$_1$	3		2				5
A$_2$			4			2	6
A$_3$		2					2
A$_4$	1	2		2	4		9
	4	4	6	2	4	2	22

There is thus a large number of feasible solutions. Which solution we choose depends on the cost of transportation. We naturally want to find that solution which gives the minimum cost.

Suppose the costs of transportation per wagon are given by

	B$_1$	B$_2$	B$_3$	B$_4$	B$_5$	B$_6$
A$_1$	90	120	90	60	90	100
A$_2$	70	30	70	70	50	50
A$_3$	60	50	90	110	30	110
A$_4$	60	80	110	20	20	110

We can find the total cost for each feasible solution and then find the solution for which the cost is minimum. It is however obvious that with this direct method, even a simple problem may take months to solve. Mathematicians have however found and proved a rule which gives us the minimum cost solution in a matter of minutes. We are not giving this rule however here.

Experiment 487. Zero-sum Two-person Game.

A and B play a game. A can make 4 moves and B can make 5 moves and for each pair of moves, the amounts to be paid by B to A are given by the following "pay-off matrix".

$$\begin{array}{cc|ccccc} & & & & B & & \\ & & 1 & 2 & 3 & 4 & 5 \\ \hline & 1 & 9 & 3 & 1 & 8 & 0 \\ A & 2 & 6 & 5 & 4 & 6 & 0 \\ & 3 & 2 & 4 & 3 & 3 & 8 \\ & 4 & 5 & 6 & 2 & 2 & 1 \end{array}$$

A wants to get the maximum amount, B wants to pay the minimum amount. What respective moves will they make ?

Let A be allowed to make the move first.

If A makes move 1, B will make move 5, so as to pay the minimum amount 0.

If A makes move 2, B will make move 3 so as to pay the minimum amount 4.

If A makes move 3, B will make move 1 so as to pay the minimum amount 2.

If A makes move 4, B makes move 5 so as to pay the minimum amount 1.

Thus the best move for A is 2, since in that case B has to pay the amount 4. This is the maximum amount which A can expect to get from B.

What A does is therefore this : he finds the minimum of each row and then chooses his move to correspond to that row which gives the greatest of these rows minima.

Now let B be allowed to make the move first.

If B makes move 1, A will make move 1 to get the maximum amount 9.

If B makes move 2, A will make move 4 to get the maximum amount 6.

If B makes move 3, A will make move 2 to get the maximum amount 4.

If B makes move 4, A will make move 1 to get the maximum amount 8.

If B makes move 5, A will make move 3 to get the maximum amount 8.

Since B wants to pay the minimum amount, he makes move 3.

What B does is this : he considers the maximum value in each column and chooses that column which gives the smallest of these maxima.

We find whether A starts first or B starts first. A will make move 2, B will make move 3 and B will pay to A an amount 4.

If A makes any other move, B can choose his move to pay less than 4.

If B makes any other move, A can choose his move to get more than 4.

We say that the value of this game is 4. Now consider the game.

		B		
		1	2	3
	1	4	7	6
A	2	5	3	7
	3	8	9	2

The maximum of row minima =4
The minimum of column maxima=7
These two do not agree.

If A plays first he chooses move 1 and ensures that he gets at least 4.

If B plays first, he chooses move 3 and ensures that he has to pay not more than 7.

It will always be found that for any rectangular array, max. (min. of rows) $\leqslant$ min. (max. of columns).

This can be seen as follows:

max. of jth column $\geqslant$ every element of jth column

$\Rightarrow$max. of jth column $\geqslant$ element of ith row and jth column, but element of ith row and jth column $\geqslant$ minimum element of ith row, so that max. element of jth column $\geqslant$ minimum element of ith row

Even the minimum element of the L.H.S. $\geqslant$ max. element of the R.H.S. so that the above result is proved.

This is an important rule.

If we arrange a number of persons in a rectangular array and measure the lengths or weights or incomes, it will always be found that

$$\text{min. (max. of columns)} \geqslant \text{max. (min. of rows)}$$

The game described is obviously a two persons' game and it is called zero-sum, since here the gain of one party is the loss of the other so that the total gain is zero.

The study of such games has important applications in economics and defence. In economics, the two parties are producers and consumers and the producers make their moves to get the maximum values for their products and the consumers make their moves to pay the minimum price they can. In war, both sides want to inflict the maximum casualties on the other side. The theory of games has also been applied to sociology and politics and in this theory, mathematics plays an important role.

Experiment 488. Inventory Control.

A businessman orders equal amounts of goods at equal intervals of times throughout the year. If he orders a large amout every time, he has to place very few orders in a year and his ordering cost is small. However he has to carry large stocks and his storage cost is high. If he orders a small amount every time, he has to place a large number of orders in a year and his ordering cost is high, but his storage cost is low.

The problem is to find how much he should order and at what interval of time so that the total of ordering and storage costs is minimum. We assume that the demand is uniform and is R units per unit time.

Suppose the businessman orders q units at a time. His stock will be exhausted in $\frac{q}{R}$ time. His stock in this interval $\frac{q}{R}$ varies from q in the beginning to 0 at the end so that his average stock is $\frac{1}{2}q$ and this stock he keeps for time $\frac{q}{R}$. The cost of storage depends both on the amount and the time. Let it be C_1 per unit per unit time. Then the cost of storage in this time $\frac{q}{R}$ is $\frac{1}{2}q.\frac{q}{R}C_1$. Any order is sufficient to meet the demands for time $\frac{q}{R}$, so that in one year the number of orders is $\frac{R}{q}$ and the total cost of storage is $\frac{1}{2}q.\ \frac{q}{R}C_1.\frac{R}{q}=\frac{1}{2}q\,C_1$.

Let C_2 be the cost per order. This cost is supposed to be independent of the order size. The cost for $\frac{R}{q}$ orders is $C_2\frac{R}{q}$.

The total ordering and storage cost $\frac{1}{2}C_1q+\frac{C_2R}{q}$. This depends on q and will change with q. However the total cost$=\frac{1}{2}C_1q+C_2\frac{R}{q}$

$$=\left(\sqrt{\tfrac{1}{2}C_1q}-\sqrt{C_2\frac{R}{q}}\right)^2+2\sqrt{\tfrac{1}{2}C_1q}\sqrt{C_2\frac{R}{q}}$$

$$=\sqrt{2C_1C_2R}+\left(\sqrt{\tfrac{1}{2}C_1q}-\sqrt{C_2\frac{R}{q}}\right)^2$$

Whatever be q, the second quantity on the right is non-negative and its minimum value is zero. Thus the total cost is minimum when

$$\sqrt{\tfrac{1}{2}C_1q}-\sqrt{\frac{C_2R}{q}}=0$$

or when $$q=\sqrt{\frac{2C_2R}{C_1}}$$

The minimum cost $$=\sqrt{2C_1C_2R}$$

The ordering interval $$=\frac{q}{R}=\sqrt{\frac{2C_2}{C_1R}}$$

Thus by simple reasoning, we have found that the businessman should order $\sqrt{\frac{2C_2R}{C_1}}$ units at a time and he should place orders at interval $\sqrt{\frac{2C_2}{C_1R}}$. If he uses this policy, his cost will be $\sqrt{2C_1C_2R}$. If he adopts any other policy, his cost will be greater.

Mathematics is used to solve much more complicated problems of stock control. By using mathematics, we can reduce the costs to a minimum and make the profits maximum.

Experiment 489. Linear Programming.

A coal merchant has two depots D_1 and D_2 which contain stocks of 30 tons and 15 tons of coal respectively. Three customers C_1, C_2, C_3 place orders for 20 tons, 75 tons and 10 tons of coal respectively. The distances from C_1, C_2, C_3 to D_1 are 3, 4, 1 miles respectively and the distances from C_1, C_2, C_3 to D_2 are 2, 1, 5 miles respectively. The cost of transporting coal is a fixed amount per ton per mile.

The problem is to find how much coal should be transported from D_1 and D_2 to each of the customers so that the total cost is minimized.

This will be recognized as a transportation problem which we shall solve by a technique known as linear programming.

We represent the data as follows :

	C_1	C_2	C_3	Available
D_1	3	4	1	30
D_2	2	1	5	15
Required	20	15	10	45

Suppose x tons are transported from D_1 to C_1 and y tons are transported from D_1 to C_2, then $30-x-y$ will be transported from D_1 to C_3, $20-x$ tons will be transported from D_2 to C_1, $15-y$ tons will be transported from D_2 to C_2. The total cost of transportation will be

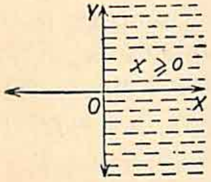
$$K[3x+4y+1(30-x-y)+2(20-x)+1(15-y)+5(x+y-20)]$$
$$=K[5x+7y-15]$$

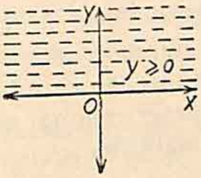
where K is the cost of transporting one ton of coal for one mile. We have to choose x and y so that this cost is minimum. However there are certain restrictions on x and y, since none of the quantities transported can be negative, it must be $\geqslant 0$. Thus we have

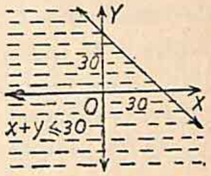
$$x \geqslant 0,\ y \geqslant 0,\ 30-x-y \geqslant 0,\ 20-x \geqslant 0,\ 15-y \geqslant 0,\ x+y-20 \geqslant 0$$

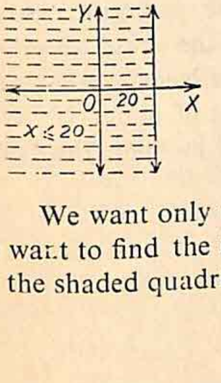
Our problem is to minimize $K(5x+7y-15)$ or $5x+7y-15$, subject to above inequalities being satisfied.

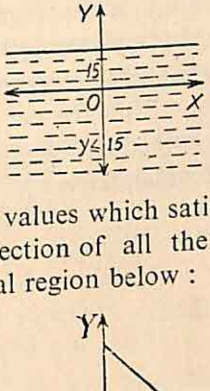
We can have a geometrical interpretation for each of these inequalities in the Cartesian plane.

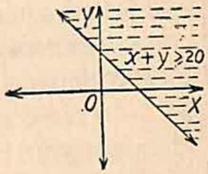




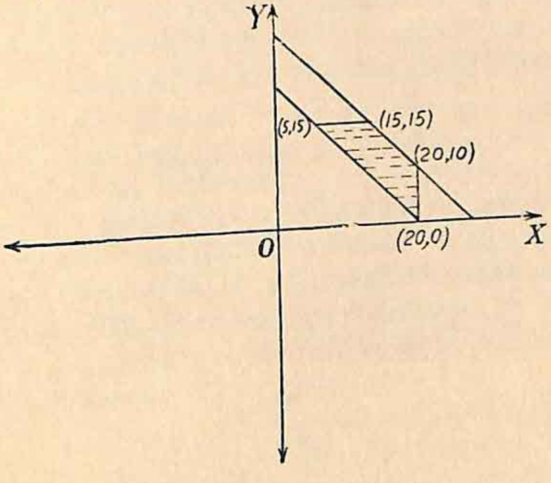








We want only those values which satisfy all the inequalities *i.e.*, we want to find the intersection of all the above sets. This is given by the shaded quadrilateral region below :



It is proved rigorously by using mathematics and can be seen intuitively that the minimum of $5x+7y-15$ must occur at a vertex of this region.

when $x=20$, $y=0$, $5x+7y-15=85$
when $x=20$, $y=10$, $5x+7y-15=155$
when $x=15$, $y=15$, $5x+7y-15=165$
when $x=5$, $y=15$, $5x+7y-15=125$

The minimum value occurs when $x=20$, $y=0$. This determines the amounts to be transported for minimum cost. Linear programming refers to the problem of maximization or minimization subject to linear constraints. The general linear programming problem for two variables is to maximize or minimize $ax+by+c$ subject to $x\geqslant 0$, $y\geqslant 0$ and $a_i x+b_i . y+c_i\geqslant 0$ and this can be solved geometrically as above. For solving general linear programming problems with a number of variables we require higher mathematics.

Linear programming problems arise in a large variety of situations and the fastest computers are used to solve these problems. About 100 crore of rupees are spent every year all over the world in solving linear programming problems and this expenditure leads to a saving of at least 1000 crore of rupees. Thus the net saving of about 900 crore of rupees per year can be ascribed to the power of mathematics ! We discuss this power in the next chapter.

21

The Power of Mathematics

Experiments : 490 to 495

Mathematics is a great intellectual enterprise and it has also tremendous applications. Children must realise the power of mathematics. They must also realise that mathematics is still growing very fast and there are still many unsolved problems in mathematics. We give below a few typical cases to illustrate the power and limitations of mathematics. The literature contains thousands of such examples.

Experiment 490. Square root of 2.

We have many standard algorithms for finding square root of 2 and we can calculate it correct to any number of places of decimals by following any one of these algorithms. A question naturally arises whether the process of finding square root will ever terminate or whether it will give rise to a recurring decimal or whether it will not terminate. This is an intellectual problem and man, as an intellectual being, cannot be happy unless he has solved the problem.

There are two methods to answer the problem. One is the "brute-force method" and the other is the "brain-force method". In the first case we go on calculating $\sqrt{2}$ to thousand, ten thousand, hundred thousand decimal places. We find no termination and no recurrence. Does it prove that there will be no termination and no recurrence ? No, because for all we know the termination or recurrence may occur at the billionth or the trillionth place. By the brute-force method, we shall go on and on and will never be able to answer this simple looking problem.

However using mathematics, we prove that

(*i*) $\sqrt{2} \neq \frac{p}{q}$ where p, q are integers and $q \neq 0$. (*Experiment 376*)

(*ii*) A number can be expressed as a terminating or recurring decimal if and only if it is a rational number. (*Experiment 376*)

These two results together prove that the process of evaluating $\sqrt{2}$ will not terminate nor it will show any recurrence.

A problem which could not be solved by 'brute power' has been solved by 'mathematical power'.

What can we say about $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{10}$, $\sqrt{11}$, ...?

We have an infinity of problems.

By the brute power method each of this infinity of problems would require an infinite amount of time. However we prove one general theorem that $\sqrt[R]{N/D}$ can be rational number if and only if N and D are perfect Rth power and we have again solved the infinity of problems by using 'mathematical power'. (*Experiment 376*)

Experiment 491. Number or Prime Numbers

We know that 2, 3, 5, 7, 11, 13, 17, ...are prime numbers. The problem is to find whether this process of finding prime numbers will terminate and we shall have only a finite number of prime numbers or the process will never terminate and we shall have an infinity of prime numbers. If $p(x)$ denotes the number of prime numbers $\leqslant x$, we have by actual counting :

$p(2)=1$, $p(3)=2$, $p(4)=2$, $p(5)=3$, $p(10)=4$, $p(100)=25$, $p(1000)=168$, $p(10{,}000)=1229$, $p(10^8)=5{,}761{,}455$ $p(10^{10})=455{,}052{,}512$.

However these calculations do not prove that the process of finding prime numbers will not terminate. It is still possible that after a certain large number, there may be no more prime numbers. No direct computation, even with the fastest electronic computers, will ever prove that the number of prime number is infinite.

On the other hand, a simple mathematical argument given more than 2,000 years ago by Euclid proves conclusively that the number of prime numbers is infinite.

If possible, let P be the largest prime number *i.e.*, all numbers greater than P are composite numbers and as such are products of prime numbers. Now consider the number

$$N=(2.\ 3.\ 5.\ 7......P)+1$$

i.e.. N is one more the product of all the finite number of prime numbers. This number is not divisible by 2, 3, 5, 7, ...or any other prime number and therefore this must be a prime number. This contradicts the assumption that P is the largest prime number. Thus there can be no largest prime number and the number of prime numbers is infinite.

However the largest known prime number is $2^{4423}-1$. This number consists of 1332 digits. This was found in 1961 with the help of an IBM 7090 computer. We do not know a larger prime number, but the above argument shows that an infinity of larger prime numbers exists.

Again this example shows the superiority of the 'mathematical power' over the 'brute power'.

A related problem is to find the number of prime numbers less than a given number. It has been shown that a very good approximate to $p(n)$ is given by

$$f(n)=\frac{n}{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{n}}$$

e.g., $p(1000)=168$, $f(1000)=167.1$.

This 'prime number theorem' is itself a great achievement of mathematical power.

Experiment 492. A Divergent Series.

Consider $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\dots+\frac{1}{n}$

As n becomes larger and larger, will the sum become greater than 10,000, will it become greater than 1,000,000, will it become greater than 10^{20}, will it become greater than 10^{100} ?

If we have a fast computer, we can answer in principle, all these questions in a finite amount of time, though in the last case, the time may be of the order of 10^{80} years ! The world cannot afford to spend even 10 years for this question.

Mathematically, however, we can show that the sum will become greater than any number we care to name. We write the series as

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\dots+\frac{1}{16}\right)+\dots$$

and see that the sum is greater than

$$1+\frac{1}{2}+\frac{2}{4}+\frac{4}{8}+\frac{8}{16}+\ \ldots\ldots$$

i.e., the sum is greater than

$$1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ \ldots\ldots$$

and so by taking a sufficiently large number of terms, we can make the sum as large as we please.

Experiment 493. Assignment Problem.

Suppose we have n men and n jobs and productivities of different men for different jobs are different. We have to assign the jobs to the men in such a way that the total production is maximized. The first job can be assigned to any of the n men in n different ways. Corresponding to each way of assigning the first job, there are $n-1$ ways of assigning the second job to any of the remaining $n-1$ men. Again corresponding to each of the $n(n-1)$ ways of assigning the first two jobs, there are $n-2$ ways of assigning the third job to any of the remaining $n-2$ men. Assigning in the same way, we find that the total number of ways of assigning n men to n jobs is

$$n\times(n-1)(n-2)\ldots\ldots 1 \text{ or } |\underline{n}.$$

In principle, we can find the total productivity for each of the $|\underline{n}$ ways and thus find that assignment which maximizes the total productivity. However for large n, this is easier said than done.

Let the children find the values of 1, 1×2, $1\times2\times3$, $1\times2\times3\times4$, $1\times2\times3\times4\times5,\ldots\ldots$, $1\times2\times3\times\ldots\times30$ and let them find the times taken if one calculation of one assignment with a fast electronic computer takes ·00001 second only. They can prepare the following table

n	5	10	12	14	16	18	20	30
$|\underline{n}\simeq$	10^2	3.10^6	5.10^8	9.10^{10}	2.10^{13}	6.10^{15}	2.10^{18}	3.10^{32}
Time	0·01 sec.	5 men	10 hours	3 months	60 years	20,000 years	10^7 years	10^{21} years

Thus it is easily seen that to solve one assignment problem for 30 men and 30 jobs would take more time the age of the Earth. However we have left out of account the 'mathematical power'. With its aid, mathematicians have developed a technique which can solve the problem in a matter of hours, if not of minutes.

In many problems, we have to solve a number of simultaneous equations. To find the truth set of n equations in n numbers, we have to perform about n calculations. From the above table, it appears that the time required would be prohibitive even if $n=20$ or $n=30$. However with the help of mathematical insight, solution of equation involve 30 or 40 unknowns is a matter of routine and in many situations, the case involving a thousand equations with a thousand unknowns have been solved.

Experiment 494. Power of Mathematics in Industry, Social Science and Astronomy.

Suppose ten different policies are available for an industrial organisation. The direct method is to try each of these ten policies and find out which is best. This is not possible, even in principle. The effect of the first policy is to change the whole structure of the industrial organisation and as such the second policy cannot be tried on the original situation.

On the other hand if we can make a mathematical model, we can find the effect of each of the ten policies on the original model and then we can choose the best policy. This is one reason why mathematical methods are of increasing importance in economics, psychology, sociology, industry and business. Another reason is that mathematical methods, whenever they are applicable, are relatively cheaper since costly experimentation is avoided or reduced.

Thus if we want to manufacture a new product, its performance may depend on 10 parameters and each may have to be tested at 10 levels. Thus 100 prototype models will have to be made to find the best model. If we can use mathematics, we may be able to find the best product without any experimentation or at least with substantially reduced experimentation.

If the parameters vary continuously, we shall have an infinity of possible products and finding the best becomes impossible, even in principle, without the use of further mathematics.

In many situations, we have to take decisions in the face of uncertainty. We have to test hypotheses when we are given only samples and we have to make predictions about the populations from which samples are drawn. We have to design experiments to get the maximum possible information with a given effort. In all these situations, mathematics is a powerful tool.

We want to know what is happening inside the Sun and the stars. We cannot get there and perform experiments. We get light from the Sun or the stars and we can analyse it in the laboratory. The only method is to make mathematical models of the Sun and the stars and deduce consequences from these. The model of which the consequences agree most closely with observations will give the most satisfactory information about what is happening inside the Sun or the stars.

Experiment 495. Limitations of Mathematics.

We have given a small sample to indicate the power of mathematics both at the intellectual and the applicational level. The students who study mathematics, science, technology, economics, biology, etc., will meet literally thousands of situations where mathematics will be found to be a powerful and indispensable tool.

We must however be also conscious of the limitations of mathematics. In spite of its vast achievements, there are literally thousands of problems which have not been solved *e.g.*, the following :

(*i*) A given map has to be coloured in such as way that adjacent countries have to be coloured differently. How many colours would be sufficient ? For all maps that have been drawn, it has been found that four colours are always sufficient, but it has yet not been possible to prove it, though the result has been proved for maps with less than 38 regions.

(*ii*) In the infinite decimal expression for π, does 1234567 occur any where ?

(*iii*) Is there an infinity of twin primes like

$(3, 5), (11, 13), (17, 19), \ldots\ldots$

where twin primes are prime number pairs differing by 2 only ?

(*iv*) Make a satisfactory model of the working of the human brain.

(*v*) Solve the problem of predicting the weather with almost complete certainty.

It is the thousands of unsolved problems both in mathematics and its applications which provide the motivation for the vigorous research in mathematics. This research is so vigorous that mathematics doubles itself in a period of about 10 years. We have to-day about 100 times as much mathematics as we had in the beginning of this century. Each problem solved gives rise to ten other unsolved problems, and so while the number of solved problems rises very quickly, the number of unsolved problems rises still faster and the number of persons required to solve these problems has to increase equally fast.

22

Puzzles and Problems

Experiments : 496 to 500

Puzzles and challenging problems provide interesting material for developing thinking power among students. Students must learn how to analyse a problem into what is given, what has to be proved and what facts can be used in solving the problem. We give below some puzzles and problems whose solution does not require any knowledge beyond high school mathematics. We also give solutions and hints for some of them.

Experiment 496. Puzzles and Problems (1—10)

1. A bottle and a cork cost 1 rupee and 10 paise and the bottle costs 1 rupee more than the cork. What is the price of the bottle ?
2. How many squares are there on a chess board ?
3. A monkey climbs up a pole of height 12′. The monkey climbs 3 feet in one hour and then slips back 2′. In how many hours does the monkey reach the top ?
4. Suppose you have a sheet $\frac{1}{1000}$th of an inch thick. Tear the sheet into two halves and put them on the top of each other. Repeat the process 50 times. What is the final thickness you get ?
5. A clock strikes 'six' in 5 seconds. How long does it take to strike 'twelve' ?
6. Prove that $3^{10}-1$ is a multiple of 8.

7. Replace letters by numbers so as to get correct sums

```
(i)    S E N D         (ii)   T H R E E
     + M O R E                T H R E E
     ---------              + F O U R
     M O N E Y              -----------
                            E L E V E N
```

8. There are 27 coins, one of which is heavier than the others. How can the faulty one be detected by weighing coins against one another on a pair of scales? What is the least number of weighings required?
9. Mark four points on a paper so that there are exactly two different distances between them.
10. You have two vessels of capacity 7 litres and 4 litres. How will you use these to get 6 litres or 2 litres of water?

Experiment 497. Puzzles and Problems (11-20)

11. In a school, there are four houses. In the inter-house competition, each house is to play against the other. How many matches are there? How many different possible results are there to the competition?
12. If 5 cats catch 5 mice in 5 minutes, how many cats are required to catch 100 mice in 100 minutes?
13. Is the number eleven thousand eleven hundred and eleven divisible by 3?
14. From a drawer, containing brown and black socks, which is the minimum number to be drawn so to ensure at least one pair of the same colour?
15. Complete the sum, given that each of the whole numbers from 0 to 9 occurs once in the digits that are included in the divisor and the quotient together.

```
          *****
       -----------
1****| 1*****1**1
       1***1*
       ------
        ***11
        *****
       -------
         1*****
          *****
         ------
          ****1
          ****1
         ------
            0
```

16. Ram told Ahmed : "In exactly 7 years, the sum of our ages will be 63". Ahmed told Ram "When you were as old as I am now, you were twice as old as I was then". What are their ages?

17. A person owns a wolf, a goat and a cabbage and he has to take these across a river in a boat which can accommodate only one more beside himself. If left alone, the wolf will eat the goat and similarly if the goat is left alone with the cabbage, the goat will eat it. How should the crossing be done?

18. How often and when will the hour-hand and the minute hand of a clock coincide?

19. If it takes a cyclist six minutes to ride a mile against the wind and only four minutes to return with the wind at his back, how much time will he take to ride a mile on a close day?

20. A man travels 100 miles at 20 miles per hour and then travels another 100 miles at 25 miles per hour. What is his average speed?

Experiment 498. Puzzles and Problems (21 – 30)

21. A man goes up a hill at 2 m.p.h. and comes down at 6 m.p.h. The total time taken is 6 hours. What is the distance up the hill?

22. A rather long number ends with an 8. If this digit is shifted to the first position, we obtain twice the original number. What is the number?

23. Find a multi-digit number which may be divided by 3 by simply shifting the last digit to the first.

24. What three-digit numbers, increased by 11 times the sum of their digits yield their own inversions?

25. In the following addition of two positive and one negative numbers, the letters have to be replaced by numerals 1 to 9 so that the sum is 100 : $abc + def - ghi = 100$. Find the numerals.

26. If we delete the common numeral occurring in the numerator and denominator of $\frac{26}{65}$, we get the correct answer. What other fractions consisting of two digit numbers in the numerator and denominator can be similarly reduced?

27. Arrange Ace, King, Queen and Jack of all four suits of a pack of cards in a square in such a way that neither a suite nor a value occurs twice in any of the four horizontal or vertical rows.

28. How can six identical match sticks be arranged to form four equilateral triangles ?

29. Show how, with a single cut, you could cut a cube of butter so that the newly cut face would be in the shape of a (*a*) a triangle (*b*) a trapezium (*c*) a rectangle (*d*) a hexagon (*f*) a pentagon.

30. The numbers 1 to 27 can be placed in the cells of a cube so that the sum of numbers in every row, every column, every file and in each of the four diagonals is the same. The solution is partly given below. Complete it.

4		26
	25	
27		10

	9	
9		19
	21	

18		1
	3	
2		24

Experiment 499. Puzzles and Problems (31-40)

31. A vessel contains 24 litres of milk which is to be shared equally by three persons. They have only three containers with capacities 5, 11, 13 litres respectively. How can they use these containers to obtain equal shares ?

32. Write a three digit number and write it again alongside to get a six-digit number (*e.g.* 456,456). Show that it is divisible by 7, 11 and 13. Explain your result.

33. Find the values of *a*, *b*, *c*, *d*, *e*, *f*, so that the following sentences may be true.

$$\begin{array}{r} +100_5 \\ +100_a \\ \hline 100_b \end{array} \qquad \begin{array}{r} 100_c \\ +100_d \\ \hline 100_5 \end{array} \qquad \begin{array}{r} 30_5 \\ 20_e \\ \hline 30_f \end{array}$$

34. A room is 30 feet long, 12 feet wide and 12 feet high. In the middle of one end wall, a foot from the ceiling, is a spider. In the middle of the opposite wall, a foot from the floor is a fly. The spider catches the fly, who is too frightened to move, by crawling all the way to him. Find the shortest route that the spider can take ?

35. In the above example, what is the shortest distance if the spider is in one corner of the roof and the fly is in the opposite corner of the floor ? Explain also how you would find

the shortest distance between two points on a circular cylinder or on a circular cone or on a pyramid or on a prism.

36. Consider

69	345	186	872	756
366	642	582	278	558
168	246	87	575	657
762	147	285	377	954
663	543	483	179	855
564	48	384	674	459

Take 5 members, one from each column. The last two digits of the sum are obtained by adding the last digits of the numbers. The first two digits are obtained by subtracting the sum from 50. Verify and explain and construct similar other examples.

37. Consider the following square of numbers

161	265	362	464	563
176	280	377	479	578
190	294	391	493	592
203	307	404	506	605
212	316	413	515	614

Choose any one of the 25 numbers and strike off its row and column, then choose any one of the remaining 16 numbers and strike off its row and column, then choose any one of the remaining 9 numbers and strike off its row and column, then choose any one of the remaining 4 numbers and strike off its row and column so that only one number remains to be chosen. Add these five numbers. Verify that the sum is 1952. Explain the reason and construct a square which gives the sum 1969.

38. The first decade consists of numbers from 1 to 10, the second decade consists of numbers from 11 to 20 and so on. Show that no decade consists of more than 4 prime numbers. Show that in the first 20 decades, there are four with 4 prime numbers each. There are six others between 81st and 85th, 116th and 120th, 186th and 190th, 206th and 210th, 326th and 330th, 346th and 350th. Find these.

39. Make 1000, using only eight 8's ; make 100, using only four 7's, make 20, using two 3's, make 100 with four 9's,

make 5 with three 5's. In each case, you have to use the numbers and the arithmetic signs only.

40. A man has eight coins, one of which is defective and heavier than the rest. Given a good pair of scales, how can you find the bad coin in only two weighings ?

Experiment 500. Puzzles and Problems (41—50)

41. If a steel band were stretched tight round the earth at the equator, so that it touches it every where, how much would the band have to be increased in length if it were placed one foot above the earth all the way round ? Assume the earth to be level at the equator.

42. A man takes 15 minutes to give a news to three persons. Each of these 3 men rushes out and relates the news to 3 persons in another 15 minutes. Each of these persons repeats the process and so on. How many persons will have news after $2\frac{1}{2}$ hours ?

43. In a certain process, we have to carry out $\underline{|n}$ mutiplications. Show that for $n=20$, the time taken by a fast electronic computer which can perform 10,000 multiplications per second would be of the order of 10^7 years.

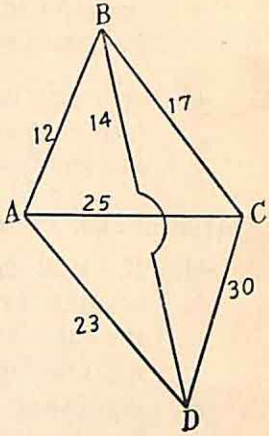
44. Consider breeding of rabbits. Each pair of rabbits produces a new pair of rabbits each month and a pair becomes productive at the age of two months. Show that the number of pairs at the end of one, two, three, four,... months is given by the Fibonnaci sequence 1,1,2,3,5,8,13,...

45. Find the relation between the square of a term of Fibonnaci sequence and the terms immediately preceding and following it. What can you say about the sum of square of two consecutive Fibonnaci numbers ?

46. Take the reciprocal of the eleventh term of a Fibonnaci sequence and express it in terms of Fibonnaci numbers alone.

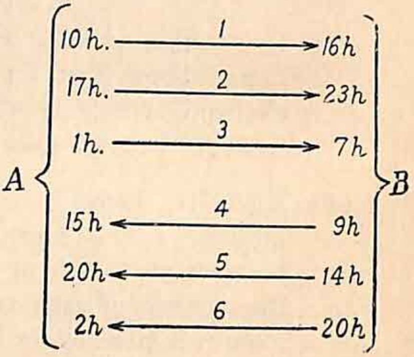
47. A travelling salesman has to start from A, stop at centres B, C, D (in any order) and come back to A. The distances

are given in the following figure. Find the shortest path for him.

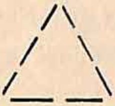
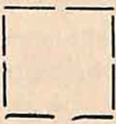
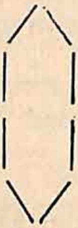
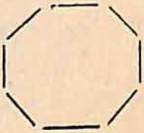


48. The time-table of an airline between two cities A and B is given below. Flight 1 leaves A at 10h (*i.e.* at 10 A.M.) and reaches B at 16h (*i.e.* at 4 P.M.), while flight 4 leaves B at 9h (*i.e.* at 9 A.M.) and reaches A at 15h *i.e.* at 3 P.M.

The pilots can live either at A or at B. Make an arrangement so that the pilots have to stay the minimum time away from home.



49. There are eight matchsticks each 1″ long. Find the area of the following figure and find which is largest.

Can you generalise this problem ?

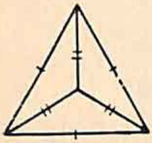
50. Take a string 12″ long. Place it on a graph paper in different shapes *viz.* triangle, square, rectangle, hexagon, circle etc. and find the area enclosed. Which gives the largest area ?

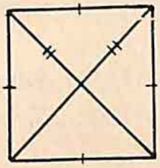
HINTS AND SOLUTIONS

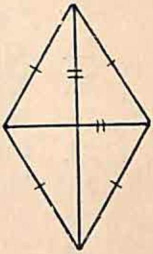
1. 105 paise 2. 204 3. 10 hours
4. Approximately 18 million miles 5. 11 seconds
6. $3^{10} \equiv 9^5 \equiv 1^5 \equiv 1$ | mod 8 |
7. $\begin{array}{r} 9567 \\ +1085 \\ \hline 10652 \end{array} \quad \begin{array}{r} 74611 \\ 74611 \\ +\ 2096 \\ \hline 151318 \end{array} \quad \begin{array}{r} 84011 \\ 84011 \\ +\ 3590 \\ \hline 171612 \end{array}$

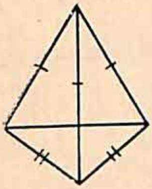
8. Split into 3 equal groups, weigh one group against another and repeat the process

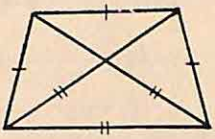
9.

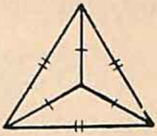




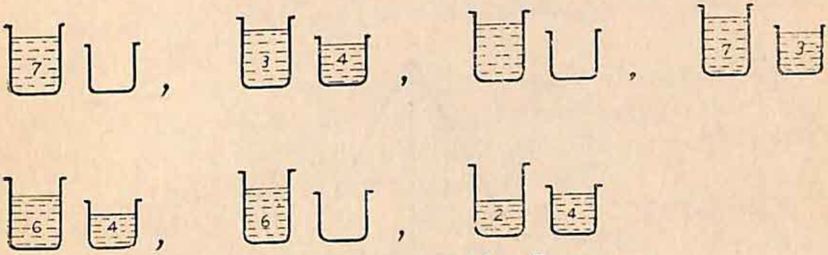








10.



11. 6729 12. 5
13. Yes, since the number is 12,111
14. 3

15.

```
             84063
15927| 1338871401
       127416
        64711
        63708
        100340
         95562
          47781
          47781
            ×
```

16. $29\frac{2}{5}$, $19\frac{3}{5}$ years.

17. Goat across, back alone, cabbage across, goat back, wolf across, back alone, goat across.

18. At twelve O'clock and there after every 1 hour $5\frac{5}{11}$ minutes, altogether 11 times in all.

19. $4\frac{4}{5}$ minutes   20. $22\frac{2}{9}$ m.p.h.   21. 9 miles

22. 421, 0, 52, 631, 578, 947, 368

23. 428, 571 or 857, 142

24. 162, 243, 324, 403, 567, 648, 729

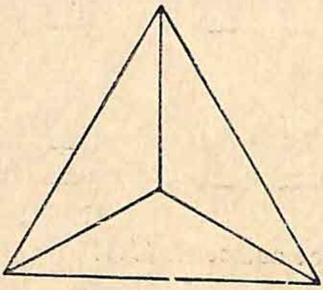
25. $458+321-679=100$; $257+189-346=100$

26. $\frac{16}{64}$, $\frac{19}{95}$, $\frac{49}{98}$

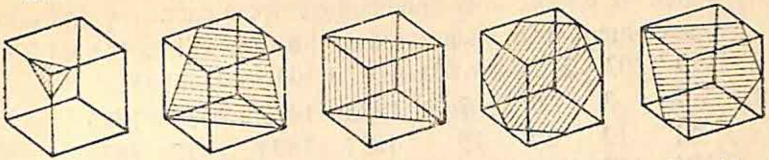
27.

A ♡	K ◇	Q ♧	J ♤
Q ♤	J ♧	A ◇	K ♡
K ♧	A ♤	J ♡	Q ◇
J ◇	Q ♡	K ♤	A ♧

28.

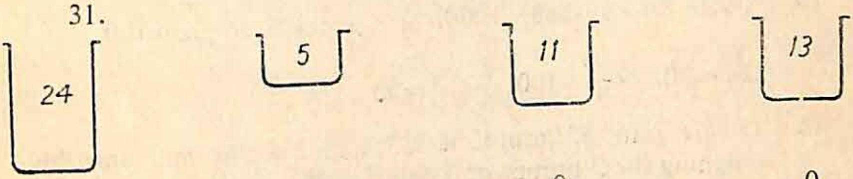


29.



30. The answer can be obtained by remembering that the sum of every row, column etc. is $\frac{1+2+3+\ldots+27}{9}=42$.

31.

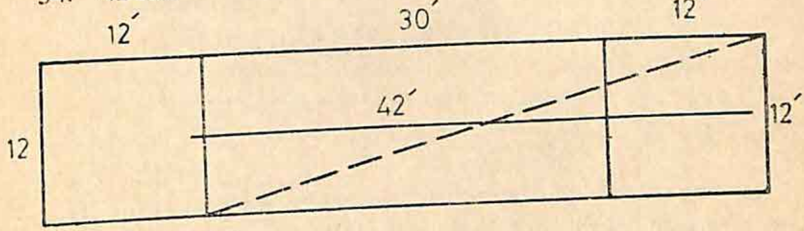


24	0	0	0
11	0	0	13
6	5	0	13
6	5	11	2
8	5	11	0
8	0	3	13
8	5	3	8
8	0	8	8

32. $7\times11\times13=1{,}001$

33. $a=12$, $b=13$, $c=3$, $d=4$, $e=12$, $f=13$

34. 42 feet



35. $\sqrt{42^2+12^2}$

36. In each column, the middle digit is the same. The sum of the middle digits is 30. In each column, the sum of the first and the last digit is the same for all numbers and the sum of these for the five columns is 47.

37. 161+0 161+104 161+201 161+303 161+402
176+0 176+104 176+201 176+303 176+402
190+0 190+104 190+201 190+303 190+402
203+0 203+104 203+201 203+303 203+402
212+0 212+104 212+201 212+303 212+402

When we choose only one element from each row and only one element from each column, the sum will be $161+176+190+203+212+0+104+201+303+402=1952$.

38.

2	3	5	7	1483	1483	1487	1489
11	13	17	19	1871	1873	1877	1879
101	103	107	100	2081	2083	2087	2089
191	193	197	199	3251	3253	3257	3259
821	823	827	829	3461	3463	3467	3469

39. $8+8+8+88+888=1000$, $\frac{7}{\cdot7}\times\frac{7}{\cdot7}=100$ or $\frac{77}{\cdot77}=100$

$\frac{\lfloor 3}{\cdot3}=20$, $99\frac{9}{9}=100$, $\frac{5}{5}\times5=5$.

40. Divide into 3 groups of 3, 3 and 2 coins and start by weighing the 2 groups of 3 coins each.

41. Only $2\pi=6{\cdot}28$ feet longer.

42. $1+3+3^2+3^3+\ldots\ldots\ldots+3^{10}=\frac{3^{11}-1}{3-1}=88{,}573$.

48. Two pilots live at B and their schedule is *a*2 and *b*3 and one pilot lives at A and his schedule is 1*b*.

49. Octagon.

50. Circle.

EXERCISES*

1. (*i*) Consider the system of numbers
   $\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, \ldots\ldots\}$
   Does every number have a successor ? Does every number have a predecessor ? If n^* is the successor of n, show that
   $n^* = n + \frac{1}{2}$, $n + m^* = (n+m)^*$, $n \times m^* = n \times m + \frac{1}{2}n$
   (*ii*) Discuss the system of numbers
   $\ldots\ldots, a-2d, a-d, a+d, a+2d, \ldots\ldots$
   from this point of view.
2. Give some systems of numbers which are closed for addition and multiplication, but are not closed for subtraction and division.
3. Give some systems of numbers in which commutative law holds for addition and multiplication but does not hold for subtraction and division.
4. Construct examples suggested in experiment 304 (*iii*) and (*iv*).
5. (*i*) Can $a + b \times c = (a+b) \times (a+c)$ be true for any natural numbers a, b, c ?
   (*ii*) Illustrate the distributive law geometrically.
6. In the system {2, 4, 6, 8, 10, ...}, do relations $>$ and $<$ satisfy trichotomy and transitive laws ? What can you say about the relations $\geqslant$ and $\leqslant$? Construct other similar examples.
7. In the above system of numbers, is the order structure consistent with both addition and multiplication structures ?
8. Do the following cancellation laws hold in N ?
   (*i*) $c + a = c + b \Rightarrow a = b$
   (*ii*) $c \times a = c \times b \Rightarrow a = b$
   Do they hold in the system of exercise 6 ?

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* In general, exercise number n refers to experiment number $300 + n$.

9. Give some systems of numbers which obey the well-ordering principle.
10. Discuss the structure of the system of whole numbers in details.
11. (*i*) Give some systems of numbers which are closed for addition, multiplication, and subtraction but are not closed for division.
    (*ii*) Give some systems of numbers which are closed for multiplication only.
    (*iii*) Give a system of numbers which is closed for addition only.
12. Do commutative, associative and distributive laws hold in the following system :—
    $\{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$.
13. Discuss the existence of the order structure and its consistency with addition, multiplication and subtraction structures for the system
    $\{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$.
14. Do the cancellation laws hold for the above systems ?
15. Compare the structures in N, I and W.
16. The system of natural numbers does not give an additive group. Why ? Extend the system by 'creating' zero and negative integers so that it forms an additive group ? In the same way extend the system I of integers so that, as far as possible, we may get a multiplicative group.
17. Consider the set of ordered pairs of natural numbers like (a, b), (c, d), in which $a, b, c, d, \ldots$ are natural numbers
    (*i*) Define $(a, b)=(c, d)$ if $a=c, b=d$. Show that this is an equivalence relation. How many members belong to an equivalence class ?
    (*ii*) Define $(a, b)=(c, d)$ if $a+d=b+c$. Show that it is an equivalence relation. Show that $(a, b)=(a+k, b+k)$ where k is any natural number. Write all ordered pairs equivalent to $(1, 2)$. Does (a, b) have properties similar to the integer $a-b$?
    (*iii*) Define $(a, b)=(c, d)$ if $ab=cd$. Does it give an equivalence relation ?
    (*iv*) (a, b) stands for the fraction $\frac{a}{b}\cdot$ Simplify
    $[(3, 4)\times(5, 6)]\div[(2, 3)\div(3, 4)]+[(3, 5)+(4, 6)]-$

[(2. 3)−(3, 2)] in the ordered pair notation itself.

(*v*) (a, b) stands for the integer $a-b$. Simplify the above expression.

18. Show that there is an infinity of fractions between 1 and 2. Show that there is an infinity of rational numbers between any two rational numbers.

19. Consider ordered pairs of integers (a, b) where a, b are integers and $b \neq 0$. Define $(a, b)=(c, d)$ if $ad=bc$. Does this give an equivalence relation ? What are the equivalence classes ? Do these give all the rational numbers ?

20. Verify all the results of experiment 320. Discuss also the possibilities

$a/b+c/d=ac/bd$ or ad/bc or $\frac{a+d}{b+c}$.

21. Give some sub-groups of the additive group of rational numbers *i.e.* give some subsets of the set of rational numbers which have a group structure for the operation of addition.

22. Try various possible definitions of $\frac{a}{b} \times \frac{c}{d}$ and show that except one, all of them do not satisfy the necessary conditions.

23. Show that the set of non-zero rationals and the set of positive rational numbers determine two multiplicative groups.

24. (*i*) Prove that the $(I, +, \times)$ is a commutative ring with unity.

    (*ii*) Prove that the set
    $\{\ldots, -3k, -2k, -1, 0, k, 2k, 3k, \ldots\}$
    where k is any positive integer, determines a commutative ring.

    (*iii*) Prove that the system $(\{0, 1, 2, 3, 4, 5\}, +_6, \times_6)$ where $+_6$ and $\times_6$ mean addition and multiplication modulo 6, also is a commutative ring in which the product of two elements can be zero without either being 0.

    (*iv*) Examine also the system $(\{0, 1, 2, 3, 4\}, +_5, \times_5)$ and generalise your result.

25. Does the set of positive rational numbers have an order structure ?

26. Are all the rational numbers between $\frac{a}{b}$ and $\frac{c}{d}$ of the form $\left[m.\left(\frac{a}{b}\right)+n\left(\frac{c}{d}\right)\right]/(m+n)$ where m and n are natural numbers? Do we get all rational numbers, if we allow m, n to be any integers such that $m+n\neq 0$? Can you enumerate all rational numbers in this way?
    (*ii*) Are N, W, I densely-pecked ?
    (*iii*) Give some subsets of Q which are densely pecked.
27. Verify all the properties stated in experiment 327.
28. Discuss the isomorphism between the set of natural numbers and a subset of integers.
29. Rewrite a chapter on fractions from any standard book on arithmetic, using the ordered-pair notation.
30. Examine the 'hexamal' and 'heptamal' representations (corresponding to decimal representation) in the scales of 6 and 7 respectively.
    Here

    $$(a_3a_2a_1 \,.\, b_1b_2b_3)_6=\left(6^2a_3+6a_2+a_1+\frac{b_1}{6}+\frac{b_2}{6^2}+\frac{b_3}{6^3}\right)_{10}$$

    In the scale of 6, verify the following :

    $\frac{1}{2}=\cdot 3,\ \frac{1}{3}=\cdot 2,\ \frac{1}{4}=\cdot 13,\ \frac{1}{5}=\cdot 1,\ \frac{1}{10}=\cdot 1,\ \frac{1}{11}=\cdot\dot{0}\dot{5}\ldots$

    Write similar expressions in the scale of 7 and deduce results similar to those of experiment 330.
31. (*i*) Discuss whether terminating 'hexamals' and 'heptamals' give commutative rings with multiplicative identity.
    (*ii*) Discuss the structure of the system of recurring decimals.
32. (*i*) Write ten non-terminating non-recurring decimals.
    (*ii*) Find the 10,000th digits in
    ·12345678910111213141516...
    and
    ·24681012141618202224.........
    (*iii*) Find the number of years in 10^N seconds.
    (*iv*) Prove from first principles that $\sqrt[3]{3}$ is an irrational number.
33. (*i*) Show that $\sqrt[3]{5}-\sqrt[4]{2}$ is an algebraic number.
    (*ii*) Find the values of e and π correct to 10 decimal places.

34. Discuss possible definitions of addition and multiplication of real numbers and make an attempt to prove that $(R, +, \times)$ is a field.

35. (*i*) Give ten sets which are countably infinite sets and prove your results.
    (*ii*) Show that the union of an enumerable set of enumerable sets is enumerable.

36. (*i*) Show that the set of irrational number is not countable.
    (*ii*) Show that the set of algebraic numbers is countable.
    (*iii*) Show that the set of transcendental numbers is not countable.

37. (*i*) Define $\sqrt{3}$, π and e according to Dedikand's approach.
    (*ii*) Verify that in cases (*a*), (*b*), (*c*) of experiments 337, the conditions (*i*), (*ii*), (*iii*) of that experiment are satisfied.

38. Give nests of intervals for determining $\sqrt{3}$, π and e.

39. Extend the algorithm given in experiment 339 to the case of cubes, fifth and seventh roots.

40. Use the algorithm given in experiment 340 to find the square root of 5, 7, 20 correct to 10 decimal places and compare the time required here with the time required by algorithm of experiment 339.

41. Discuss how the convergence of the algorithms given here depends on k and m, by considering some specific examples.

42. Extend the table given in experiment 342 to the cases of finite modular and complex number systems.

43. (*i*) Verify all the results stated in experiment 343.
    (*ii*) A binary operation may or may not be commutative, may or may not be associative, may or may not possess an identity, may or may not possess an inverse for every element. How many possibilities are there ? Try to illustrate each.

44. Give five examples from day-to-day life and five examples from mathematics where the commutative law does not hold.

45. How many possible values are there for
    (*i*) $a_1 \div a_2 \div a_3 \div a_4 \div \dots \div a_n$
    (*ii*) $a_1 \times a_2 \times a_3 \times a_4 \times \dots \times a_n$
    (*iii*) $a_1 \div a_2 \div a_3 - a_4 - a_5 - a_6 \div a_7$.

46. For N, I, R, Q, R_0, Q_0, find which of the twenty-four distributive laws hold.

47. Discuss +, −, ×, ÷ as binary mappings in N, I, Q, R. Discuss also mappings from N×Q to N, N×I to I etc.
48. Discuss all the possibilities envisaged in experiment 348.
49. Obtain the inverses of the following mappings :

$(\times_3)$ from Q_0 to Q_0
$(\div_5)$ from Q_0 to Q_0
$(+_3)$ from R to R.

50. $P \wedge Q$ (read P and Q) is a sentence which is true if and only if both sentences P and Q are true. $P \vee Q$ (read P or Q) is a sentence which is true if and only if either P or Q or both are true. Prove that

(*i*) $P \wedge Q \equiv Q \wedge P$ (*ii*) $P \vee Q \equiv Q \vee P$
(*iii*) $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$ (*iv*) $(P \vee Q) \vee R \equiv (P \vee Q) \vee R$
(*v*) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ (*vi*) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
(*vii*) $P \wedge P \equiv P$ (*viii*) $P \vee P \equiv P$.

51. (*i*) Give examples of sentences where the first sentence is true, the second is true, but when the second is true, the first need not be true.
(*ii*) Give five other examples of operations which do not lead to equivalent equations.
52. Prove that the four operations of experiment 351 do lead to equivalent sentences.
53. Find the simplest possible members of the equivalent classes of the following equations :

(*i*) $2x+3=5x+8$
(*ii*) $x+3>2x+5$
(*iii*) $x+y=2$ and $2x+3y=5$
(*iv*) $x+y+z=3,\ x+2y+3z=6,\ x+3y+5z=9$.

54. Examine whether '⇒' gives an equivalence relation. Show that the following belong to different equivalence classes :

(*i*) $x+y=1,\quad 2x+2y=3,\quad 3x+3y=5$
(*ii*) $2x+5=0,\quad x+3=0,\quad x+2=0$.

55. Verify and illustrate all the laws stated in experiment 555. Simplify the inequalities :

$x+3-2x+5>3x+6$
$x+y+2 \quad >3x+4y+6$.

56. Write notes on (*i*) importance of zero (*ii*) importance of place value system for mathematics. Discuss the statement

that modern civilization would have been impossible without these two concepts.

57. In the Greek system of numeration try to do the following problems :

(*i*) 356+244. (*ii*) 356+336.
(*iii*) 356−338. (*iv*) 21×14.
(*v*) 42÷6.

58. In the Roman system of numeration, do the following problems :

(*i*) 4567+3425 (*ii*) 4567−3425
(*iii*) 4565+5 (*iv*) 225×15.

59. In the Egyptian system of numeration, do the following problems :

(*i*) 567+345 (*ii*) 567−345
(*iii*) 565÷5 (*iv*) 565×5.

60. Write an essay on the scale of 12. Rewrite as much arithmetic as possible in this scale.

61. Write the numbers 34567 in different numerals used in India.

62. (*i*) Express all numbers from 1 to 100 in terms of four fours.
    (*ii*) Express all numbers from 1 to 50 in terms of two digits 2 and 3 (used any number of times).

63. Give more examples of the type given in experiment 363.

64. (*i*) Draw a graph showing the number of steps required to reach a symmetric number in the reversed addition process for numbers upto 100.
    (*ii*) Draw a similar graph for the modified reversed addition process.

65. (*i*) Use binary-denary system to find 4568+3456 and 4568−3456.
    (*ii*) How many different representations can you have for 543 ? Draw a graph showing the number of representations for numbers from 1 to 100.

66. How many equilateral triangles of edge 2 units are in the Chinese checkers design with edge lengths 2, 3, 4, 5, 6, 7, 8 units. Comment on the pattern obtained.

67. (*i*) Find some numbers of the form $\frac{abc}{def}$ in which cancellation of one or two of common digits from numerator and denominator gives the right answer.
 (*ii*) Compare (*iv*) of experiment 367 and experiment 491.
 (*iii*) Find next three numbers in the sequence 2, 3, 5, ... Discuss at least four distinct possibilities.
 (*iv*) Show that $(n-40)^2+(n-40)+41=n^2-79n+1601$ is prime for $o<n\leqslant 79$.

68. (*i*) Give truth sets of $\square+\triangle=6$ in N, W, I, Q, R
 (*ii*) Give truth sets of $\square+\triangle<6$ in N, W, I, Q, R
 (*iii*) Find the solution set of $x^n-1=0$ in the system $(\{0, 1, 2, ..., p-1\}, +_p, \times_p)$ for different values of n and p.
 (*iv*) If a quadratic equation has an irrational root $a+\sqrt{b}$, it must also have a root $a-\sqrt{b}$. Is it always true ?

69. Solve the following equations by using mappings or graphs
 (*i*) $3x=9$ (*ii*) $x=\frac{3}{x}$ (*iii*) $x^2+x+1=0$ (*iv*) $x^3-x+1=0$.

70. (*i*) Moke factor trees for 480, 920, 1155 and verify unique factorization theorem.
 (*ii*) Does unique factorization theorem hold in I or Q ?

71. (*a*) Prove
 (*i*) $(n_1, n_2, n_3, ...)\ (m_1, m_2, m_3, ...)=(n_1+m_1, n_2+m_2, ...)$
 (*ii*) $(n_1, n_2, n_3, ...)\div(n_1, m_2, m_3, ...)=(n_1-m_1, n_2-m_2, ...)$
 (*b*) Show that every positive rational number can be written as $(n_1, n_2, n_3, ...)$, where $n_1, n_2\ n_3, ...$ are integers.
 (*c*) Show that a rational number $(n_1, n_2, n_3, n_4, ...)$ in which all components except n_1, n_3 are non-negative integers, admits of a finite decimal representation.

72. Prove that H.C.F. of $(a_i, b_i, c_i, ...), (i=1, 2, ..., n)$ is $(\min a_i, \min b_i, \min c_i, ...)$ and the L.C.M. of these numbers is given by $(\max a_i, \max b_i, \max c_i, ...)$.

73. Show that the above rules for finding H.C.F. and L.C.M. remain true when $a_i, b_i, c_i, ...$are any integers.

74. Show that for any two fractions, the product of their H.C.F. and L.C.M. is equal to the product of the two fractions.

75. Find the H.C.F. and L.C.M. of 35, 42, 49, 56, 63, 70 by using the usual algorithm for finding L.C.M. and justify your result.

76. Justify rigorously the statement made in the last paragraph of experiment 376. Give independent proofs of irrational nature of $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt[3]{3}$ etc. by using unique factorization theorem.
77. Find the multiplier cycles for 19, 23, 31, 37.
78. Find the remainders when the following numbers are divided by 7, 11, 13, 29 respectively :
    56789345, 2345678231, 1005670089003.
79. Since 7.11.13=1001, show that if a number is divisible by 7 or 11 or 13, then that number reduced by 1001, 2002, ... or 9009, or 10010, ..., or 900900 is also divisible by any of these. Use this result to show that 479563 is divisible by 7 and not by 11 or 13.
80. (*i*) Prove that a number is divisible by 2^n if and only if the number represented by the first n digits from the right is divisible by 2^n.
    (*ii*) Prove that a number is divisible by 5^n if and only if the number represented by the first n digits from the right is divisible by 5^n.
    (*iii*) Use the results $10^n \equiv 1$, (mod 9) to obtain the tests of divisibility by 3 and 9.
    (*iv*) Use the result $10 \equiv -1 \pmod{11}$, $10^{2n} \equiv 1 \pmod{11}$, to find the test of divisibility by 11.
81. Develop criteria for divisibility by 2, 3, 4, 5, 6, 7, 11, 12, 13 in the scales of 7 and 8. Develop also criteria corresponding to that of experiment 381 in these scales.
82. Give ten polynomials in n which are even for all values of n and ten polynomials which are odd for all values of n.
83. Give ten polynomials in n of the following types :
    (*i*) divisible by p for all values of n
    (*ii*) leaving remainders 1, 2, 3, ..., $p-1$ when divided by p where p is a prime number.
84. Let u_n denote the nth term of the Fibonacci sequence
    1, 1, 2, 3, 5, 8, 13, 21,......
    and v_n denotes the nth term of the allied sequence due to Lucas
    1, 3, 4, 7, 11, 18, 29, 47, ...
    then verify the following :
    (*i*) u_n, u_{n+1} are coprime *i.e.*, have no common factor except 1.

(*ii*) v_n, v_{n+1} are also co-prime.

(*iii*) u_n and v_n are both odd or both even and in the second case H.C.F. of u_n and v_n is 2.

(*iv*) u_n is a divisor of u_{nr} for every r.

(*v*) If m and n are co-prime, then u_m and u_n are co-prime.

(*vi*) If d is the H.C.F. of m and n, then u_d is the H.C.F. of u_m and u_n.

(*vii*) $u^2_n - u_{n-1}u_{n+1} = (-1)^{n-1}$.

(*viii*) $v^2_n - v_{n-1}\, v_{n+1} = (-1)^n 5$.

(*ix*) $v^2_n - 5u^2_n = (-1)^n 4$.

(*x*) $u_m v_n + u_n v_m = 2u_{m+n}$.

Try to establish some of these results.

85. Find the H.C.F. of 28, 70, 84 and justify the method.
86. Discuss the existence and uniqueness of H.C.F. and L.C.M. in the domain of polynomials with integral coefficients.
87. Discuss the existence and uniqueness of H.C.F. and L.C.M. of the system $(\{0, 1, 2, \ldots, p-1\}, +_p, \times_p)$ when (*i*) p is prime number, (*ii*) p is a composite number.
88. Discuss the possibilities of existence and uniqueness of the concepts of H.C.F. and L.C.M. in the system of real numbers.
89. Find the general multinomial corresponding to
    (*i*) $1.2.3+2.3.4+3.4.5,\ 2.3.4+3.4.5+4.5.6, \ldots$
    (*ii*) $1.2.3+2.4.6+2.6.9,\ 3.4.5+6.8.10+9.12.15,\ \ldots$
90. Find the general multinomial corresponding to
    (*i*) $2.3.4+3.4.6+4.6.2,\ 3.5.7+5.7.11+7.11.3,\ \ldots$
    (*ii*) $2^1.3^2+3^4.4^5,\ 3^2.4^3+5^6.6^7,\ 5^4.6^5+4^5.5^6,\ \ldots$
91. Find the general expression corresponding to
    (*i*) $3^{5^7},\ 5^{9^{15}},\ 11^{4^7},\ 2^{1^{10}},\ \ldots$
    (*ii*) $3^5+7^{10},\ 4^8+5^9,\ 2^{20}+3^{11},\ 5^2+4^1,\ \ldots$
92. Give examples of polynomials in four or five variables.
93. Give geometrical illustrations for $ab+bc+cd$.
94. Use geometrical illustrations to show that
    (*i*) $(a+b)(c+d) = ac+ad+bc+bd$.
    (*ii*) $(x+a)^3 = x^3+3x^2a+3xa^2+a^3$.
    (*iii*) $(x+a)(x+b)(x+c) = x^3+x^2(a+b+c)+x(ab+bc+ca)+abc$
95. Explain and illustrate each of the fourteen uses of graphs mentioned in experiment 395.

96. Draw actually at least six of the graphs mentioned in experiment 396. Suggest at least six more graphs of the same type.

97. Draw graphs of multiplication tables of numbers from -10 to 10 in one large figure and verify the commutative and associative laws for multiplication from the graphs. Verify also the rule for finding the sign of the products of two integers.

98. Draw actually at least eight of the graphs mentioned in experiment 398. Suggest at least six more similar graphs.

99. Draw actually at least five of the graphs mentioned in experiment 399. Suggest at least five more similar graphs.

100. Draw actually at least five of the graphs mentioned in experiment 400. Suggest at least five more similar graphs.

101. (*i*) Suppose mathematics doubles itself in a period of 10 years. Draw a graph showing its growth from 1900 to 1970.

 (*ii*) Make a binary slide rule of your own.

102. Draw some of the graphs mentioned in experiment 402. Suggest at least five new graphs.

103. Draw graphs of truth sets of

 (*i*) $1 \leqslant |x-1| \leqslant 2$ (*ii*) $|x| > -3$ and $|x| < 1$

 (*iii*) $[x] < 5$ (*iv*) $2 < [x] \leqslant 5$

 (*v*) $[x^2-1] \geqslant 0$ (*vi*) $y = [x]$

 (*vii*) $y = [x-1] + [x-2] + [x-3]$

 (*viii*) $y = |x-1| + |x-2| + |x-3|$

 (*ix*) $|x| + |y| = 2$ (*x*) $|x| + |y| = |x-y|$

104. Draw graphs of truth sets of

 (*i*) $y = |x| + [x]$ (*ii*) $y = |[x]|$

 (*iii*) max. $(x, y) > 1$ (*iv*) min. $(x, y) < 1$.

105. Construct the rectangular hyperbolic paraboloid suggested in experiment 405.

106. Draw the graphs of amounts against time for all the national certificates Draw graphs to compare the yields with investments at 4 percent, 5 percent, 6 percent and compound interests graphically.

107. Find graphically the time at which 5 percent compound interest and 7 percent simple interest will lead to the same amount.

108. (*i*) Draw the income-tax curves for 1967-68, 1968-69, 1969-70.
  (*ii*) Draw the fare-distance graph for railway fares.
  (*iii*) Draw some of the graphs for mensuration formulae.

109. (*i*) Give ten examples of congruent objects.
  (*ii*) Give ten examples of objects which are similar but not congruent.

110. Obtain the conditions for the congruence of circles, triangles and polygons.

111. Obtain the conditions for the congruence of spheres, right circular cylinders, right circular cones and right pyramids.

112. Take some pairs of congruent figures in the plane and show how by translation, reflection and rotation, one member of each pair may be made to coincide with the other member of the pair.

113. What are the equivalence classes obtained by the congruence relation ? Is similarity mapping an equivalence relation ? Does it always determine a one-one mapping ?

114. (*i*) A mapping preserves distances AB, BC, CD, DA between four non-collinear points. Is it a congruence mapping ?
  (*ii*) A mapping maps the point (x, y) onto the point $(ax+b, cy+d)$. What is the condition that it is a congruence mapping ? How many congruence mappings are there ? What do you call them ?
  (*iii*) A mapping maps the point (x, y) onto the point $(ax+by, cx+dy)$. What is the condition that it is a congruence mapping ? How many such congruence mappings will be there ? What do you call them ?

115. Find all congruence mappings which leave points (a, b), (c, d) fixed. Do they form a group ?

116. Find all congruence mappings which take the point (a, b) to (c, d) and $(a+r\cos\alpha,\ b+r\sin\alpha)$ to $(c+r\cos\beta,\ d+r\sin\beta)$. Do they form a group ?

117. Find all congruence mappings which take the points (a, b), $(a+r\cos\alpha,\ b+r\sin\alpha)$, $(a+r\cos\beta,\ b+r\sin\beta)$ respectively to
(c, d), $(c+r\cos(\alpha+\theta),\ d+r\sin(\alpha+\theta))$,
$(c+r\cos(\beta+\theta),\ d+r\sin(\beta+\theta))$.
How many such congruence mappings are there ?

118. A reflection in the axis of x maps point (x, y) onto point $(x-y)$. Use this to prove analytically the eight results of experiment 418.
119. Find axes of symmetry for letters of alphabet of all the languages of India.
120. A rotation through an angle α about the origin maps the point (x, y) onto the point $(x \cos \alpha - y \sin \alpha,\ x \sin \alpha + y \cos \alpha)$. Use this to prove analytically all the eight properties of experiment 420.
121. Draw ten figures which do not change by rotation through suitable angles about some suitable points.
122. Find the points of central symmetry of a square, a rectangle, a parallelogram, a regular hexagon and an ellipse ? Given two congruent figures in a plane, can you always obtain one from the other by a certain rotation ? Illustrate your answer.
123. A translation maps the point (x, y) onto point $(x+a, y+b)$. State and prove analytically some properties of translation mapping.
124. Prove analytically the three results of experiment 424.
125. Prove analytically the result of experiment 425.
126. Prove analytically the result of experiment 426.
127. Prove analytically the result of experiment 427.
128. Discuss the symmetries in letters of the alphabet, in playing cards, in ornaments, in architecture, in flowers, in birds, in animals, in crystals, in polygons, in polyhedra etc.
129. Analyse all the symmetries in the designs of experiment 429.
130. Discuss the properties of all the transformations given in experiment 430.
131. Use the concept of shearing to find areas of triangles, parallelograms and trapezia.
132. Use the concept of shearing to find volumes of polyhedra, prisms, pyramids etc.
133. Find formulae for areas of circles, triangles, quadrilaterals, regular polygons by drawing them on graph paper.
134. Find formulae for areas of regular figures by cutting and pasting.
135. Take a large number of circles, cut them into a large number of triangles each, paste them on a graph paper and use these to find an approximate value of π.

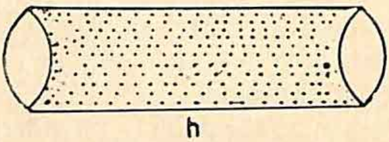
136. (*i*) Find a formula for the area of the curved surface of a cylinder when the basic curve is not a circle and all the generators are inclined at some angle (not necessary a right angle) to the plane of the curve.

(*ii*) Give an experimental method for finding the area of the curved surface of any cone.

137. Give an experimental method for finding the areas of the faces of any polyhedron.

138. Can you obtain all the results of experiment 438 by weighing solids of different dimensions ?

139. The centre of a sphere of radius r moves through a distance h.



What is the volume of the shaded portion ? Find similar expression for volumes generated when a prism or a pyramid or a circular disc moves in the same manner.

140. How many ordered triplets are solutions of

□ + △ + ▭ = n

when $n=1, 2, 3, 4, 5, 6$ and the universal set is N. Comment on the pattern. Discuss also the results when the universal set is W.

Do the same for □ + △ + ▭ ⩽ n.

141. Make a model to show the truth set of

□ + △ + ▭ = 6.

142. Discuss the truth sets of

(*i*) □ + △ + ▭ = 6, □ + 2△ + 3▭ = 14, □ − 2△ + 5▭ = 12
(*ii*) □ + △ + ▭ = 6, □ + 2△ + 3▭ = 14, 3□ + 5△ + 7▭ = 34
(*iii*) □ + △ + ▭ = 3, □ + △ + ▭ = 6, □ + △ + ▭ = 94
(*iv*) □ = 0 △ = 0 □ + △ = 4.

143. Find the number of members in the truth set of

−9 ⩽ □ = △ + ▭ ⩽ 9,

when the universal set is W.

144. Discuss the location of the positions of objects mentioned in experiment 344.

145. In the game of experiment 345, give which points each player will like to reach to get a straight line. Discuss some possible winning strategies.

146. Construct exercises similar to exercise 142 to discuss the various possibilities.

147. A line passes through (0, 0, 0), (1, 1, 1), (2, 2, 2). Find at least ten different linear relationships between x, y, z. How many of these are independent ?

148. Find the number of elements in the truth set of

□+△+▭+⬡$=n$

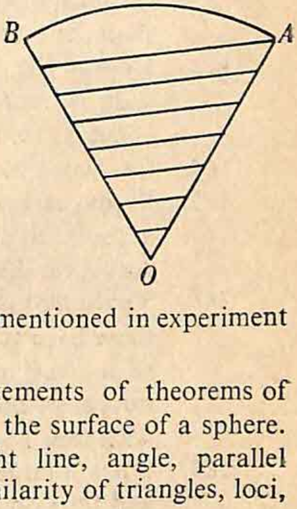
where $n=1, 2, 3, 4, 5, 6, 7,$......and the universal set is N (or W). Comment on the patterns.

149. What are $x^2+y^2=1$, $x^2+y^2=2$, $x^2+y^2=3$ etc ? Discuss the shape of $x^2+y^2=z$.

150. Find the truth set of $x^2+y^2=3z^2$ and find the semi-vertical angle of the cone it represents.

151. Find the truth set of $x^2+y^2+z^2=25$ when the universal set is W. Find the distance of every point of the truth set from the origin.

152. Cut out the figure on the right and fold it in the form of a right circular cone. Can you find the equation of this curve ?



153. Study the plane section of surfaces mentioned in experiment 453.

154. Investigate modifications in the statements of theorems of plane geometry for the geometry on the surface of a sphere. Examine also the concepts of straight line, angle, parallel lines, triangle, congruence and similarity of triangles, loci, circle theorem, etc. in this case.

155. How many great circles pass through any two points of the surface of sphere ? How many small circles pass through these ?

156. Find the shortest paths on the surfaces of the Earth between New Delhi, London, Paris, New York, Washington, Tokyo, Peking, Moscow. Compare these with the actual sea and air routes used.

157. State some results of plane geometry which do not hold for spherical geometry. Discuss the limiting case when the radius of the sphere tends to infinity.

158. Will $c^2 < a^2 + b^2$ for every right-angled triangle on the surface of the sphere? What can be the necessary and sufficient conditions for two spherical triangles to be congruent?

159. Find the longitudes and latitudes of ten most populous cities of the world. Suppose in New Delhi it is 12 o'clock noon, what will be the times in these cities?

160. Define celestial sphere, zenith, horizon, celestial equator, north polar distance, declination, hour angle, azimuth, cardinal points. From a nautical almanac, find the declination of the Sun on different days of the year.

161. Find the declinations of some stars and of moon and planets from a nautical almanac. What do you observe?

162. Trace the variation of zenith distance and azimuth of a star throughout the night. If the declination of a star is greater than the latitude of a place, will it ever set?

163. Discuss the daily motion of the Sun at a place whose latitude is more than $67\frac{1}{2}°$ as the declination of the Sun changes from $-23\frac{1}{2}°$ to $23\frac{1}{2}°$ in a year.

164. Carry out the practical work suggested in experiment 464.

165. Study, as much as possible, the geometry on the surface of a torus, a cone and a cylinder. Discuss plane sections, curves on the surface and co-ordinates.

166. Verify that if four houses are to be joined to four stations, there have to be at least five intersections. Show also that on a torus, it is possible to connect three stations with three houses without any intersection. What will be the position with four houses and four stations on a torus?

167. Draw a large number of networks with no odd vertices, two and four odd vertices, and verify Euler's theorem for these. Can a graph have an odd number of odd vertices?

168. Give a general rule as to the road network in a city, if one-way traffic has to be allowed on all roads without disconnecting any two points. How will you find the minimum

number of roads on which two-way traffic should be permitted ?

169. Draw a directed graph and interpret it as giving results of a league match system.

170. Draw some graphs representing equivalence relations and some graphs which are transitive but are not reflexive or symmetric.

171. For any map drawn on a plane or a sphere, verify that four colours are always sufficient. Draw a map on a torus requiring seven colours.

172. Make all the five regular polyhedra in the way suggested in experiment 472.

173. Find restrictions for hexagonal and octagonal pyramids on the same lines as in experiment 473.

174. Make a rhombo-dodacahedron.

175. Make the models suggested in experiment 475.

176. Answer all the questions raised in (*ii*) of experiment 476. With three dice, find the probability of the sum of points being 11.

177. Complete all the tables of experiment 477.

178. (*a*) In a very hotly fought battle, 70 percent at least of the combatants lost an eye, 75 percent at least lost an ear, 80 percent at least lost an arm and 85 percent at least lost a leg. How many at least must have lost all four ?

(*b*) A market investigator returns the following data : Of 1000 people consulted, 811 liked chocolates, 752 liked toffee and 418 liked boiled sweets, 570 liked chocolates and toffee, 356 liked chocolates and boiled sweets,, 348 liked toffees and boiled sweets and 297 liked all the three. Show that this information as it stands must be incorrect ?

179. From a box containing a white and b black balls, a ball is drawn n times without replacement. Find the probability that r of these are white where $r=0, 1, 2, \ldots\ldots, n$.

180. In experiment 480, is it necessary that OM=ON ? If $OM=\frac{1}{2}ON$, show that PN=2PM.

181. In experiment 481, suppose it is not possible to measure EF. Suggest how you would determine H.

182. (*i*) Suggest a method for determining the direction for constructing a straight road between two points, a few miles apart on plane ground.

(*ii*) Discuss the problem of determining the direction of digging of an under-sea tunnel between two specified points on French and British coasts or on Indian and Ceylonese coasts.

183. Can you suggest a method for measuring the dimension of the Earth from measurements made from a satellite ?

184. Suggest some other problems in which properties of triangles can be used to determine heights and distances of inaccessible objects.

185. A book binder has one printing press, one binding machine and the manuscripts of 6 different books. The time required in days to perform printing and binding operations of each one are given below. In how many different ways he can do it and what will be the total time taken for each way ?

Book	Printing time	Binding Time
1	3	8
2	12	10
3	5	9
4	2	6
5	9	3
6	11	1

Can you find the sequence which will take the least time ?

186. Give some feasible solutions of the following transportation problems and find their costs :

	B_1	B_2	B_3	B_4	B_5	Available
A_1	2	11	10	3	9	4
A_2	1	4	7	2	1	8
A_3	3	9	4	8	12	9
Required	3	3	4	5	6	21

187. (*a*) Find the value of the game

$$\begin{array}{cccc} & & B & \\ & 7 & 3 & 8 \\ A & 6 & 5 & 7 \\ & 9 & 4 & 10 \end{array}$$

(*b*) Write five 4×4 matrices at random and verify in each case that

$$\text{min. (max. of columns)} \geqslant \text{max. (min. of rows)}.$$

188. Find the optimal order quantity, using model of experiment 488, for the case where the demand is 4000 units per month, cost is Rs. 12 per unit, ordering cost is Rs. 100 per order and the carrying cost is Rs. 12 per unit per year.

189. A factory makes cricket bats and tennis rackets. A cricket bat takes 1·5 hours of machine time and 2 hours of craftsman's time, while a tennis racket takes 2·5 hours of machine time and 1·5 hours of craftsman's time. In a day, the factory has available upto 80 hours of machine time and 70 hours of craftsman's time.

(*a*) What is the largest number of (*i*) bats (*ii*) rackets which could be made in a day in the factory ?

(*b*) What number of bats and rackets must be made if the factory is to work at full capacity ?

(*c*) The profit on a bat and a racket is Rs. 5 and Rs. 3·50 respectively. Find the maximum profit to the factory on a day when it produces (*i*) only bats (*ii*) only rackets and (*iii*) works at full capacity.

190. (*a*) Find $\sqrt{2}$ correct upto 5, 10, 15, 20 decimal places and estimate the times required to find it correct upto 100, 1000, 10,000 decimal places.

(*b*) Use formula $\pi=4(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\ldots\ldots)$ to find the value of π correct to 3, 5, 7, 9 decimal places and estimate the time required to get the values correct to 20, 50, 100 decimal places.

191. Verify that $p(1000)=168$.

Find $f(2)$, $f(3)$, $f(4)$, $f(5)$, $f(10)$, $f(100)$, $f(1000)$, and compare these with the values of $p(2), p(3), \ldots, p(1000)$ respectively.

192. (*a*) Show that the sum of the series

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\ldots+\frac{1}{2^n}$$

will never exceed 2, whatever n may be.

(*b*) Show that the sum of the series

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}+\ldots+\frac{1}{\sqrt{n}}$$

can be made as large as we please by choosing n to be sufficiently large.

193. Verify the values given in table of experiment 393.

194. Discuss with some statistician how mathematics helps in drawing conclusions about a population from the knowledge of a sample only. Similarly discuss with a mathematician or an astronomer or a physicist how the knowledge of mathematics helps in finding the constitution of the Sun and the stars.

195. (*a*) How many colours are required for (*i*) the political map of India (*ii*) political map of the world?

(*b*) Find first fifty twin primes.

196. (*a*) Prove that a number $(g\,f\,e\,d\,c\,h\,a)$ is divisible by 13 if $a-3b-4c-d+3e+4f+g-3h-4k\ldots\ldots$ is divisible by 13. The multiplier cycle here is $(1, -3, -4, -1, 3, 4)$. Find similar multiplier cycles for divisibility by 11, 17, 19, 23 and 31. Show that in some sense there always exists a multiplier cycle for divisibility by any number.

(*b*) Find multiplier cycles for divisibility by 11, 14, 16 in the scale of 7.

197. (*a*) Show that there is an infinity of numbers divisible by 11 which are not self numbers. Show that the result remains true in other scales.

(*b*) Show that a numeral which represents a self number in the scale of ten may not represent a self number in another scale and *vice versa*.

198. Show that a number *abcd* in which a and d are distinct and b and c are distinct will give rise, after one reversed subtraction and one subsequent reversed addition to 9999 if $c>b$ and to 10890 if $c<b$. Discuss the case when a and d or b and c are not distinct. State and prove similar results for 3, 5 and 6 digit numbers and for numbers in the scale of 7.

199. If there are 1200 students in a school, show that

(*i*) there is at least one day on which four or more students have their birth day.

(*ii*) there is at least one month in which hundred or more students have their birth day.

(*iii*) there are at least 75 students whose heights differ by one inch or less.

(*iv*) there are at least two students whose weights differ by 1 kilogram or less and whose heights differ by 2 inches or less. The heights are supposed to be between 2′ and 6′ and weights lie between 30 kg and 80 kg.

State one principle of which the above can be regarded as particular cases and give more examples.

200. You are given two sets of numbers

$$N=\{1, 2, 3, 4, 5, 6, 7, 8, \ldots\ldots\}$$
$$=\{1, 2, 4, 6, 8, 10, 12, \ldots\ldots\}$$

and the following definitions :

(*i*) A number x of a set S is a factor of y if there exists a number z of S such that $xz=y$.

(*ii*) A number is a prime number if and only if its only factors are unity and the number itself ; other numbers are called composite.

For the set E, prove the following statements :

(*a*) the number of primes in E is infinite ;

(*b*) there exists a formula for all primes in E ;

(*c*) every composite number cannot be factorized uniquely as a product of primes.

Answer these questions also for the set T where

$$T=\{1, 3, 6, 9, 12, 15, \ldots\ldots\}$$

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